SQUARING A CIRCLE AND VICE VERSA MADE EASY UP TO ANY DESIRED APPROXIMATION

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Abstract

As per the History of Indian Mathematics, there werte differences of opinion between two sects of ritual performers of the Vedic age regarding the construction of the altar/Chiti called Garhapatya that triggered the problems of squaring a circle and circling a square. In addition to this, the transcendental nature of π involved in these problems made them impossible.

Baudhayana, author of Sulva Sutra, gave a simple method to convert a square into a circle using 3.088 for the value of π. Furthermore, he gave an entirely new procedure to convert a circle into a square which can be translated mathematically as,

$$s = d\left[1 - \tfrac{1}{8} + \tfrac{1}{8\times 29} - \tfrac{1}{8\times 29\times 6} + \tfrac{1}{8\times 29\times 6\times 8}\right]$$

where s is the side of the square, and d is the diameter of the circle. In this process also the π value comes out to be 3.088. But this procedure is very cumbersome because part of the diameter of the circle has to be constructed. When the same π value is considered for the construction of both the problems, squaring a circle and circling a square, why was a single method of construction with reversible nature not adopted, since one problem is the reversal of the other?

Famous mathematicians, Srinivasa Ramanujan and Jacob de Gelder, gave their own methods to convert a circle into square. Both of them took π value as 355/113. The precision through their method is restricted to 6 digits of π only. Why cannot the precision be taken to any degree one wishes? This paper gives a method that can provide answers to the above-mentioned self-posed, potent and pertinent questions.

Key words: ritual, chiti, transcendental number, sulva sutra, π value

Introduction

A look into the history of Indian Mathematics reveals that in the days of the Vedic civilization (1500 – 500 BC), Indians felt that it was necessary to perform sacrificial rites to please gods and have boons and blessings from them. For this they needed different, specific shapes of altars or *Vedis* of definite areas so as to make their offerings acceptable to gods. Especially when they had to construct the *Garhapatya* altar, some opined that it should be of square shape while some others asserted that it must have circular shape. Consequently there arose the need for squaring a circle and vice versa, since in both the cases the area is to remain unaltered. Thus the impossible problem (by virtue of its involvement with the transcendental number π) squaring a circle is as old as the Vedas, the repositories of knowledge.

Before proceeding to the objective of this article, I intend to have a glance at the different modes of contributions of the prominent mathematicians, *Bodhayana*, the author of the oldest *Sulvasutra* of Vedic age, *Srinivasa Ramanujan*, the world renowned Indian mathematician of nineteenth century, and *Jakob De Gelder* (1849) towards the solutions for the problems squaring a circle and vice versa upto the possible extent of accuracy.

Bodhayana's Construction for Circling a Square

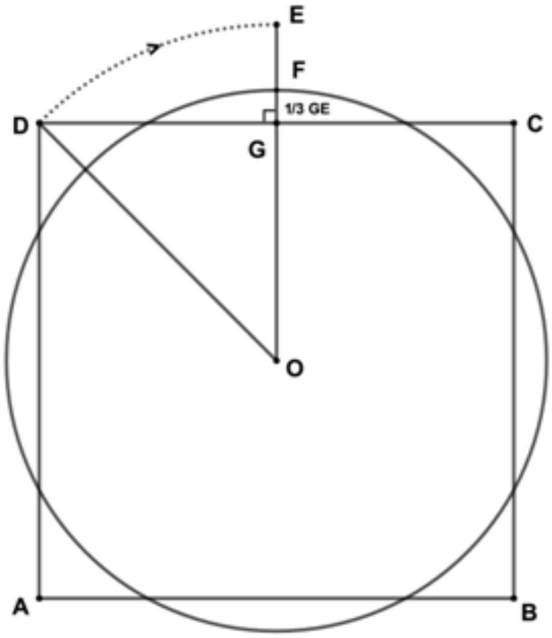

Figure 1

Construction:

1. Let ABCD be a square with center O.

2. Rotate OD till it assumes the position of OE perpendicular to DC, cutting DC in G.

3. Construct GF equal to $\frac{1}{3}$ GE.

4. With OF as radius , describe a circle which will be equal in area to the square ABCD.

On calculation we get $\pi = 3.088 = \frac{AB^2}{OF^2}$

Proof:

Let *a* be the side of the square

OD = $\frac{1}{2}$ the diagonal of the square

$= \frac{a\sqrt{2}}{2}$

$= \frac{a}{\sqrt{2}}$

$\therefore \quad OF = \frac{a}{2} + \frac{1}{3}\left[\frac{a}{\sqrt{2}} - \frac{a}{2}\right]$

$\therefore \quad 2\,OF = \frac{a}{3}(2 + \sqrt{2})$

$\therefore \; d = \frac{a}{3}(2 + \sqrt{2})$, where d is the diameter of the circle.

$\frac{a}{d} = \frac{3}{2 + \sqrt{2}}$

$\Rightarrow \; \frac{a}{d} = \frac{3(2 - \sqrt{2})}{2}$ ………………….. (1)

Now $\frac{\pi d^2}{4} = a^2$ gives

$\frac{a}{d} = \frac{\sqrt{\pi}}{2}$

$\therefore \; \frac{\sqrt{\pi}}{2} = \frac{3(2 - \sqrt{2})}{2}$

$\Rightarrow \sqrt{\pi} = 3\left[2 - \left(1 + \frac{1}{3} + \frac{1}{3\times4} - \frac{1}{3\times4\times34}\right)\right] \quad \left(\because \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\times4} - \frac{1}{3\times4\times34}\right)$

$\therefore \sqrt{\pi} = \frac{3 \times 239}{12 \times 34}$

$= \frac{514089}{166464}$

$= 3.088$

Bodhayana's Construction for Squaring a Circle

Divide the diameter into eight equal parts, and again one of these parts into twenty nine parts; of these twenty nine parts and the sixth part of the one left part less, with the eighth part of the sixth part. Since this construction becomes clumsy, it is not depicted here. In this case also π equals to 3.088.

Proof :

If a be the side of the square and d be the diameter of the circle.

$$\therefore a = d\left[1 - \frac{1}{8} + \frac{1}{8\times29} - \frac{1}{8\times29\times6} + \frac{1}{8\times2\times6\times8}\right]$$

Here again,

$$\frac{\pi d^2}{4} = a^2 \text{ gives}$$

$$\frac{\sqrt{\pi}}{2} = \frac{a}{d} = \left[1 - \frac{1}{8} + \frac{1}{8\times29} - \frac{1}{8\times29\times6} + \frac{1}{8\times29\times6\times8}\right]$$

$$\therefore \sqrt{\pi} = \frac{9785}{5568}$$

$$\text{Or } \pi = \frac{95746225}{31002624} = 3.088$$

This value is almost equal or just better than the value contained in the Indian mythological scripture, the Mahabharatha, and the Holy Bible.

Verses in Mahabharata

परिमण्डलो महाराज स्वर्भानुः श्रूयते ग्रहः ।
योजनानां सहस्राणि विष्कम्भो द्वादशास्य वै ॥४०॥
परिणाहेन षट्त्रिंशद्विपुलत्वेन चानघ ।
षष्टिमाहुः शतान्यस्य बुधाः पौराणिकास्तथा ॥४१॥
चन्द्रमास्तु सहस्राणि राजन्नेकादश स्मृतः ।
विष्कम्भेण कुरुश्रेष्ठ त्रयस्त्रिंशत्तु मण्डलम् ।

These may be translated thus:

1. *'O Great King! It is heard that the Rahu Graha is parimandala (round on all sides), and its diameter is twelve thousand yojanas.(40)*
2. *sinless king , in circumference, it is thirty six (thousand yojanas), and its thikness is said to be sixty hundred (yojanas) by experts in Puranas. (41)*
3. *king, the Moon is described to be eleven thousand(yojanas) in diameter. O Kurusrestha, the peripheral circle thereto is thirty three (thousand yojanas). O Great Soul, the Moon is fifty nine (hundred yojanas) in thickness. (42)*
4. *Kurunandana, the Sun in eight thousand plus another two (thousamd yojanas) in diameter. O King, the peripheral circle therefore is equal to thirty (thousand yojanas) . (43)*
5. *Sinless King, it is fifty eight hundred (yojanas) in thickness.*

The dimensions described above may be tabulated as follows (all figures are in yojanas):

Graha	*Diameter (D)*	*Circumference (C)*	*Thickness (t)*
Rahu	*12000*	*36000*	*6000*
Moon	*11000*	*33000*	*5900*
Sun	*10000*	*30000*	*5800*

Thus the $\frac{C}{D}$ in all three cases is 3 which, therefore, is the value of π used without any doubt.

The same value can be drawn from the verses in The Bible (I Kings 7:23)

The verses run as follows:

There He made the sea of cast bronze , ten cubits from one brim to the other, it was completely round. It's height was five cubits and a line of thirty cubits measured its circumference.

From the two constructions of Bodhayana it can be ascertained that he might have adopted different modes of constructions for the same value of π, because either one of the two constructions have no reversible facility.

Jakob De Gelder's Construction of π

In 1849, Jakob de Gelder gave an interesting construction of π using the convergent, $\frac{355}{112}$.

His construction runs as follows.

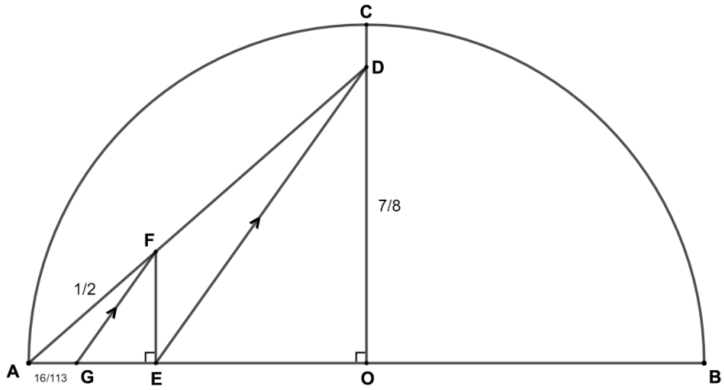

Figure 2

Let O be the center of a circle with radius OC =1 . Let AB be a diameter perpendicular to OC. Let OD = $\frac{7}{8}$ and AF = $\frac{1}{2}$. Draw FE parallel to CO and FG parallel to DE. This AG becomes equal to $\frac{16}{133}$. Then $3 \times$ OC+AG becomes equal to $3 + \frac{16}{113} = \frac{355}{113}$.

$$AD^2 = AO^2 + OD^2 = 1 + \frac{49}{64} = \frac{113}{64}$$

In the Δ DAO , FE $\parallel$ DO

$$\frac{AE}{AO} = \frac{AF}{AD}$$

$$\Rightarrow AE^2 = \frac{AF^2}{AD^2} \times AO^2 = \frac{1/4}{113/64} \times 1 = \frac{16}{113}$$

Similarly in Δ AED , FG $\parallel$ DE

$$\frac{AG}{AE} = \frac{AF}{AD}$$

$$AG^2 = \frac{AF^2}{AD^2} \times AE^2 = \frac{16}{113} \times \frac{16}{113} = \frac{16^2}{113^2}$$

$$\therefore AG = \frac{16}{113}$$

Using this method we can construct a square equal in area to the given circle. The method goes as follows.

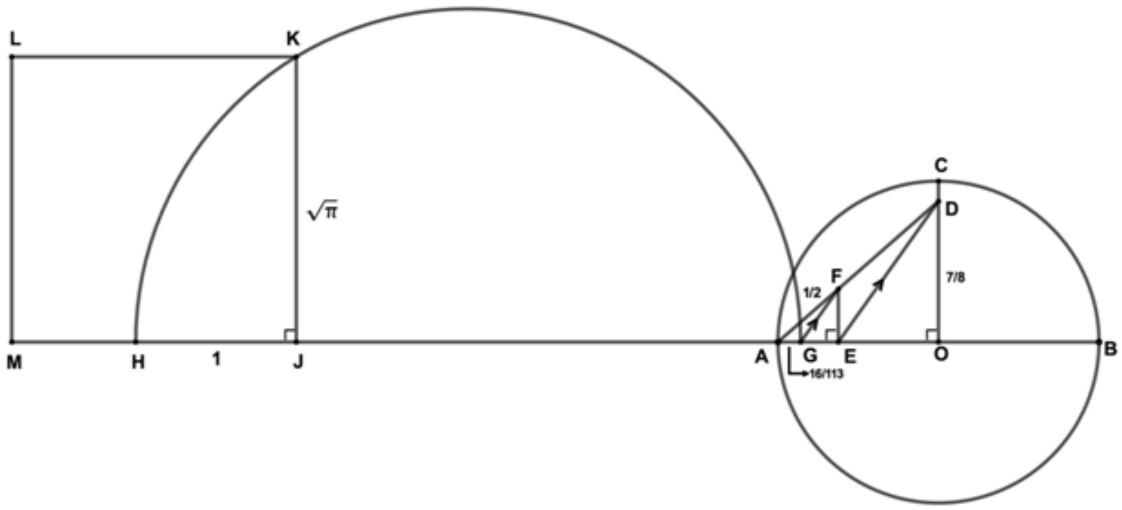

Figure 3

Construction:

1. Take the circle to be converted into a square of equal area and proceed with the above method for the line segment AG equaling to $\frac{16}{133}$

2. Extend BA to H such that AH = 4 CO

3. Draw a semi circle on HG

4. Locate J on HG such that HJ = CO and draw KJ perpendicular to HG

5. Construct the square JKLM

Now the area of JKLM = Area of the Circle with center O

$JK^2 = JG.HJ$

$JK^2 = JG.HJ^2$ (since HJ = 1)

$\therefore\ JK^2 = \pi\, r^2$

Remark:

Though this method is simple, it too doesn't provide scope to trace back the circle.

Srinivasa Ramanujan's Method for Squaring a Circle

Ramanujan's mode of solution for the problem, displaying his unmatchable ingenuity, goes as follows:

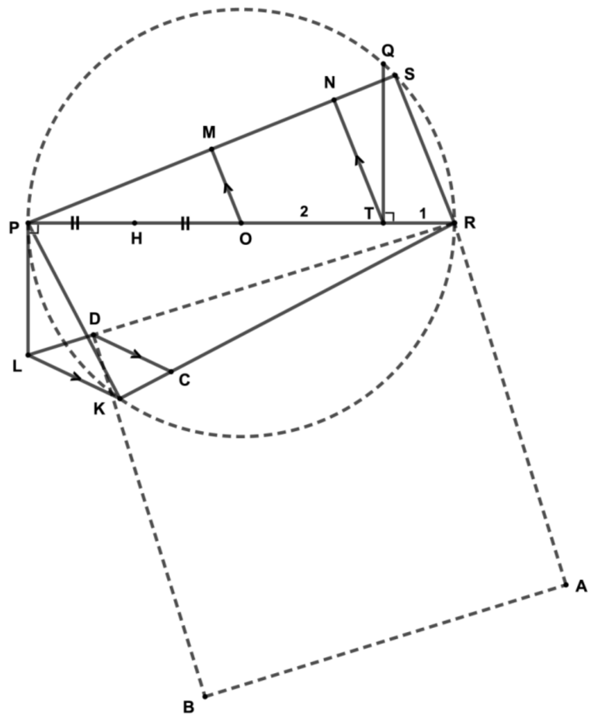

Figure 4

Construction:

1. Let O be the center and PR any diameter of the given circle.

2. Bisect OP at H and trisect OR at T (2:1)

3. Draw TQ perpendicular to OP.

4. Draw RS = TQ

5. Join PS

6. Draw OM and TN parallel to RS

7. Draw PK = PM and PL = MN and perpendicular to OP

8. Join RL, RK and KL

9. Cut off RC = RH

10. Draw CD parallel to KL

Now $RD^2 = \odot$ O.

On calculation $\pi = \frac{RD^2}{OR^2} = \frac{355}{113}$.

Proof :

Let d and r are the diameter and radius of the circle respectively

$$RS^2 = TQ^2 = PT.TR = \frac{5}{6}d \cdot \frac{1}{6}d = \frac{5}{36}d^2$$

So $PS^2 = PR^2 - RS^2$

$$= d^2 - \frac{5}{36}d^2 = \frac{31}{36}d^2$$

Since in Δ PSR , MO $\parallel$ SR

$$\frac{PM}{PS} = \frac{PO}{PR} = \frac{1}{2}$$

$$\therefore PM^2 = \frac{1}{4}PS^2 = \frac{1}{4} \times \frac{31}{36}d^2$$

$$= \frac{31}{144}d^2$$

Also $\frac{MN}{PM} = \frac{OT}{PO} = \frac{\frac{2}{3} \cdot r}{r} = \frac{2}{3}$

$$\therefore MN^2 = \frac{4}{9}PM^2$$

$$= \frac{4}{9} \cdot \frac{31}{144}d^2 = \frac{31}{324}d^2$$

Now $RL^2 = RP^2 + PL^2$

$$= d^2 + MN^2$$

$$= d^2 + \frac{31}{324}d^2$$

$$= \frac{355}{324}d^2$$

Now $RK^2 = RP^2 - PK^2$

$$= d^2 - PM^2$$

$$= d^2 - \frac{31}{144}d^2$$

$$= \frac{113}{144}d^2$$

Now Δ RCD $\sim$ Δ RKL

$$\therefore \frac{RD}{RC} = \frac{RL}{RK}$$

Hence $RD^2 = \frac{RL^2}{RK^2} \cdot RC^2$

$$= \frac{355}{324} \times \frac{144}{113} \times \frac{9}{16}d^2$$

$$= \frac{355}{113} \times \frac{d^2}{4}$$

$$\therefore RD^2 = \frac{355}{113}r^2 = \pi r^2$$

Ramunujan, it seems, did not try the reversal, i.e., circling a square.

This also does not allow us to trace back the circle from the square.

From these solutions given by Bodhayana, Jakob de Galder and Srinivasa Ramanujan, we can conclude that:

1. In order to arrive at different approximated rational values for the transcendental number π through the equation $\frac{x^2}{r^2} = \pi$, where *x* is the side of the square and *r*, the radius of the circle, Bodhayana, Jakob de Galder and Srinivasa Ramanujan adopted dufferent modes of constructions.

Question 1: Can't we have a fixed single method for obtaining any plausible value for π?

2. Another noteworthy point is that their non-reversible nature, i.e tracing back the circle from the square or vice versa, seems to be far from practicable.

Question 2: Can't we have a method which can provide feasibility for reversal?

In view of this, I suggest a single fixed method fusing the answers for the self-posed, pertinent and potent questions : Question 1 and Question 2 which are the objectives of this paper, namely,

1. Obtaining any nearest value for π through, $\frac{x^2}{r^2} = \pi$*, x being the side of the square and r, the radius of the circle.*

2. Providing the nature of reversibility in construction.

Construction for Squaring a Circle

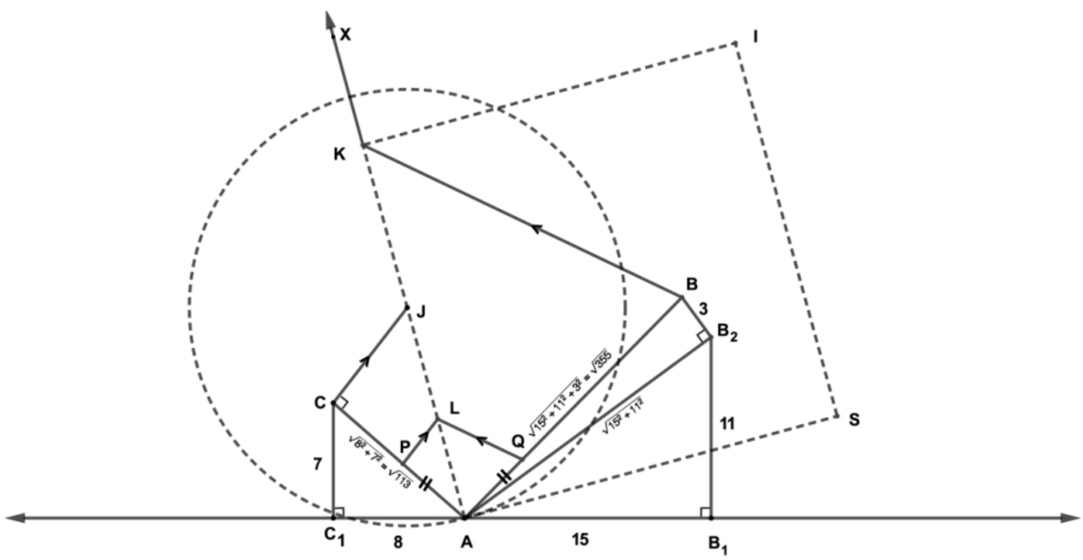

Figure 5

Construction:

1. Using Pythogorean principle construct two line segments AB of length $\sqrt{355}$ units [AB = $\sqrt{AB_1^2 + B_1B_2^2 + B_2B^2} = \sqrt{15^2+11^2+3^2} = \sqrt{355}$] and AC of length $\sqrt{113}$ units [AC = $\sqrt{AC_1^2 + C_1C_2^2} = \sqrt{8^2+7^2} = \sqrt{113}$] making any convenient angle.
2. Draw any arbitrary angle divider $\overrightarrow{AX}$ of $\angle BAC$.
3. Cut off P and Q on AC and AB respectively with any convenient length such that AP = AQ and each less than AC.
4. Locate J on $\overrightarrow{AX}$ such that AJ equal to the radius of the circle whose area is to be made equal to the area of a square.
5. Join CJ and draw PL parallel to CJ cutting $\overrightarrow{AX}$ at L.
6. Join LQ and draw BK parallel to LQ cutting $\overrightarrow{AX}$at K.
7. Now AK becomes the side of the square KASI the area of which equals to the area of the given circle with J as center and JA as radius.

Proof:

Since PL ∥ CJ

Δ APL $\sim \Delta$ ACJ

$$\frac{AC}{AP} = \frac{AJ}{AL}$$

$$\frac{\sqrt{113}}{AP} = \frac{AJ}{AL}$$

$$AJ^2 = 113.\frac{AL^2}{AP^2} \text{ .... } (1)$$

Similarly

$\because$ BK ∥ QL

Δ AQL $\sim \Delta$ ABK

$$\therefore \frac{AB}{AQ} = \frac{AK}{AL}$$

$$\frac{\sqrt{355}}{AQ} = \frac{AK}{AL}$$

$$AK^2 = 355.\frac{AL^2}{AQ^2} \text{ .... } (2)$$

On dividing (2) by (1) we get

$$\frac{AK^2}{AJ^2} = 355.\frac{\cancel{AL^2}}{\cancel{AQ^2}}.\frac{1}{133}.\frac{\cancel{AP^2}}{\cancel{AL^2}}$$

$$\frac{AK^2}{AJ^2} = \frac{355}{113} \quad (\because \text{by construction } AP^2 = AQ^2)$$

$$AK^2 = \frac{355}{113}.AJ^2$$

$$= \pi r^2 \quad [\because r = AJ]$$

Then what about circling a square?

See Fig 5

Construction:

1. Just follow the steps from step (1) to step (3) of the above construction for squaring a Circle.
2. Locate K on $\overrightarrow{AX}$ such that AK equal to the side of the given square KASI whose area is to be made equal to the area of a circle.
3. Join KB and draw QL Parallel to KB cutting $\overrightarrow{AX}$ at L
4. Join LP and draw CJ Parallel to LP cutting $\overrightarrow{AX}$ at J
5. Describe a circle with J as center and JA as radius.

Now the area of the circle is equal to the given square KASI

Proof of the previous figure holds good for this one also.

It is evident from the figure above that construction of line segments equal to the lengths of square root units of numerator and denominator related to the rational form of π plays a pivotal role in the conversion of a circle into a square or vice versa. In constructing such line segments, Lagrange's four-square theorem (any natural number can be expressed as sum of one, two, three or four squares only) comes to one's help to lessen the burden of constructing number of right triangles. We can arrive at such lengths of line segments by constructing at the maximum three right triangles one upon the hypotenuse of the other. There are several rational forms of π with respective errors, correct to one, two, three, etc., decimal places. So we can convert a square into a circle or vice versa with any desired error or desired approximation. Following this, another diagram with rational π value as $\frac{103993}{33102}$ is depicted here. This value further narrows down the difference of areas of circle and square than that of the π value $\frac{355}{113}$. To draw line segments of length $\sqrt{103993}$ units and $\sqrt{33102}$ units, $103993 = 243^2 + 212^2$ and $33102 = 91^2 + 96^2 + 118^2 + 41^2$ are considered. (There may be other ways of expressing the numbers as sum of squares)

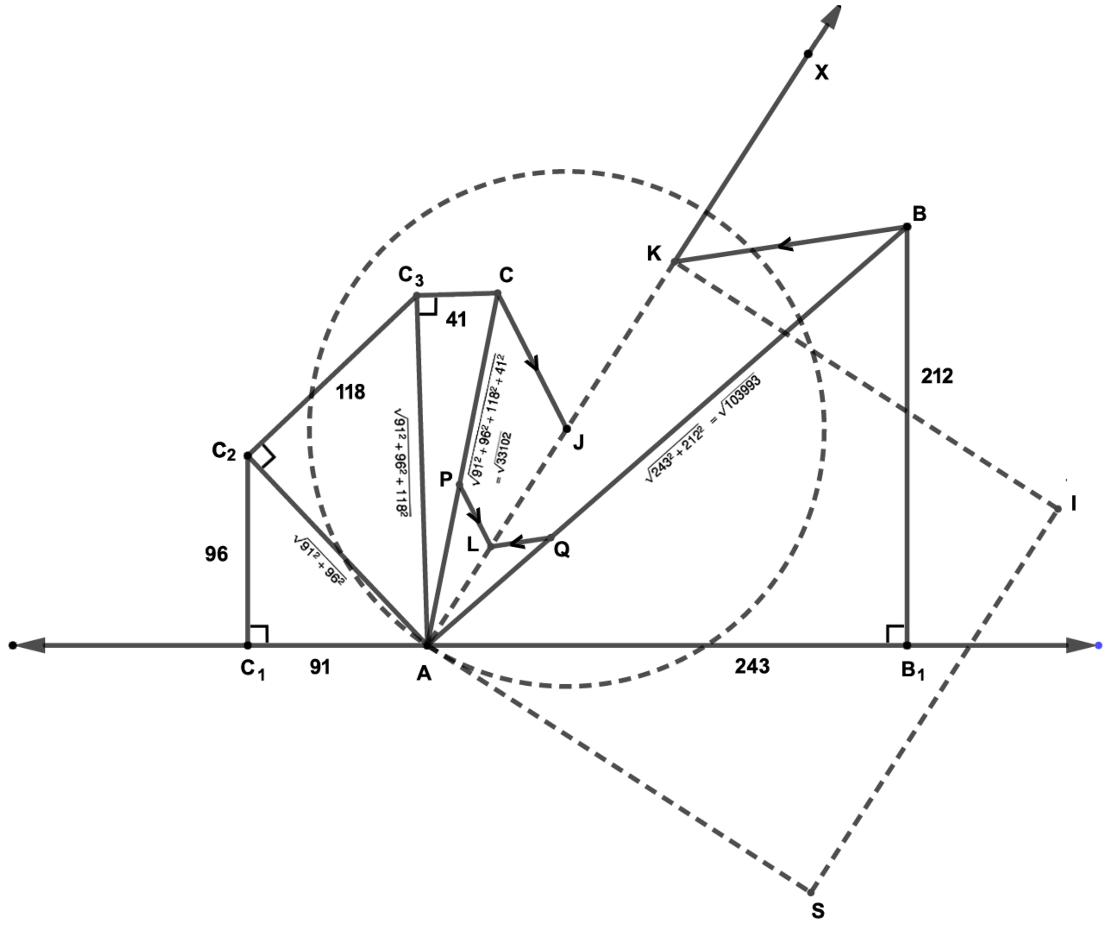

Figure 6

Proof of the previous figure holds good to this one also.

$$\text{So, } \frac{AB^2}{AC^2} = \frac{AK^2}{AJ^2}$$

$$\frac{103993}{33102} = \frac{AK^2}{AJ^2}$$

$$\pi = \frac{AK^2}{AJ^2}$$

$$\therefore AK^2 = \pi\, AJ^2$$

$$= \pi\, r^2$$

Another interesting aspect is that this method of circling a square can be extended to constructing a circle equaling the area of any given convex polygon of *n* sides with $\frac{103993}{33102}$ as the π value. Its mode of construction is depicted below.

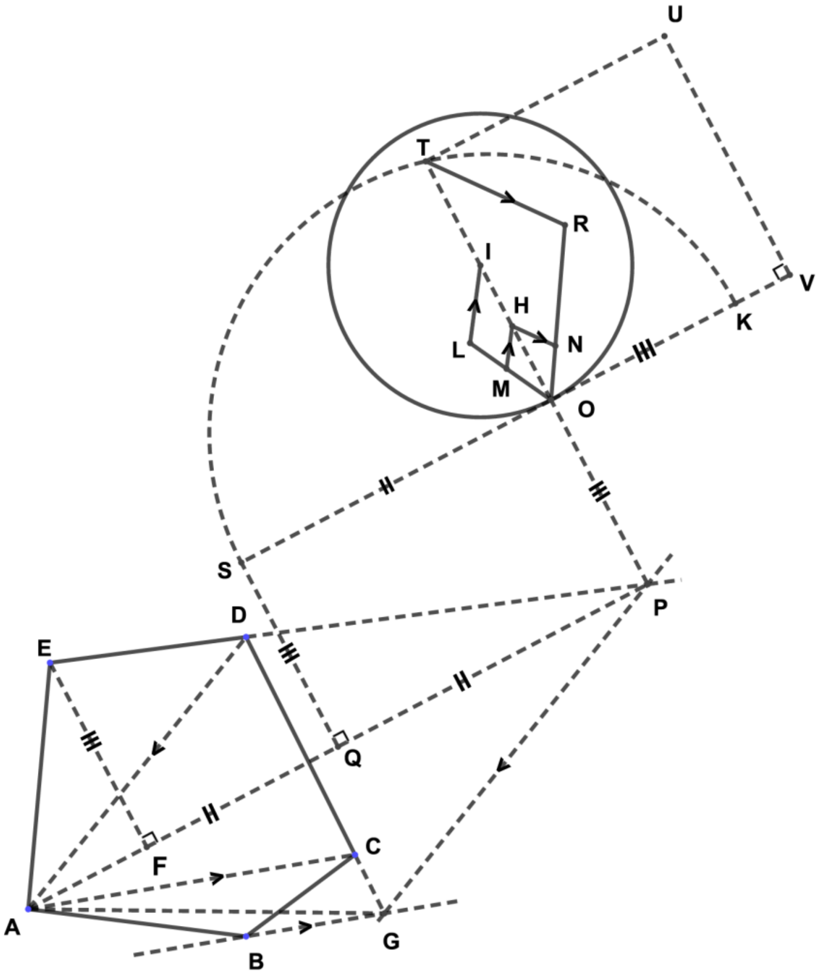

Figure 7

Polygon ABCDE $= \Delta\,AED + \Delta\,ADC + \Delta\,ABC$

$= \Delta\,AED + \Delta\,ADC + \Delta\,AGC$ $[\because \Delta\,ABC = \Delta\,AGC]$

$= \Delta\,AED + \Delta\,AGD$ $[\because \Delta\,ADC + \Delta\,AGC = \Delta\,AGD]$

$= \Delta\,AED + \Delta\,APD$ $[\because \Delta\,AGD = \Delta\,APD]$

$= \Delta\,APE$ $[\because \Delta\,APD + \Delta\,ADE = \Delta\,APE]$

$= \frac{1}{2} \,.\, AP \,.\, EF$

$= PQ \,.\, QS$ $[\because EF = QS]$

$=$ ▭ QPQS

$= \square$ QVUT

$= \odot$ I

$=$ The circle with I as center and OI as radius!

Another feature:

Squaring a circle or circling a square can be mathematically expressed as $x^2 = \pi r^2$, where x is the side of the square and r is radius of the circle. The same equation can be interpreted as *the area of the square is made equal to π times the area of another square constructed on radius of the circle with radius r.* So we can magnify or diminish a square by a rational number of times. Hence this procedure also helps to construct a circle equal to a rational number of times of the area of any given convex polygon. Here a circle is constructed making its area equal to 1.32 times the area of the pentagon ABCDE

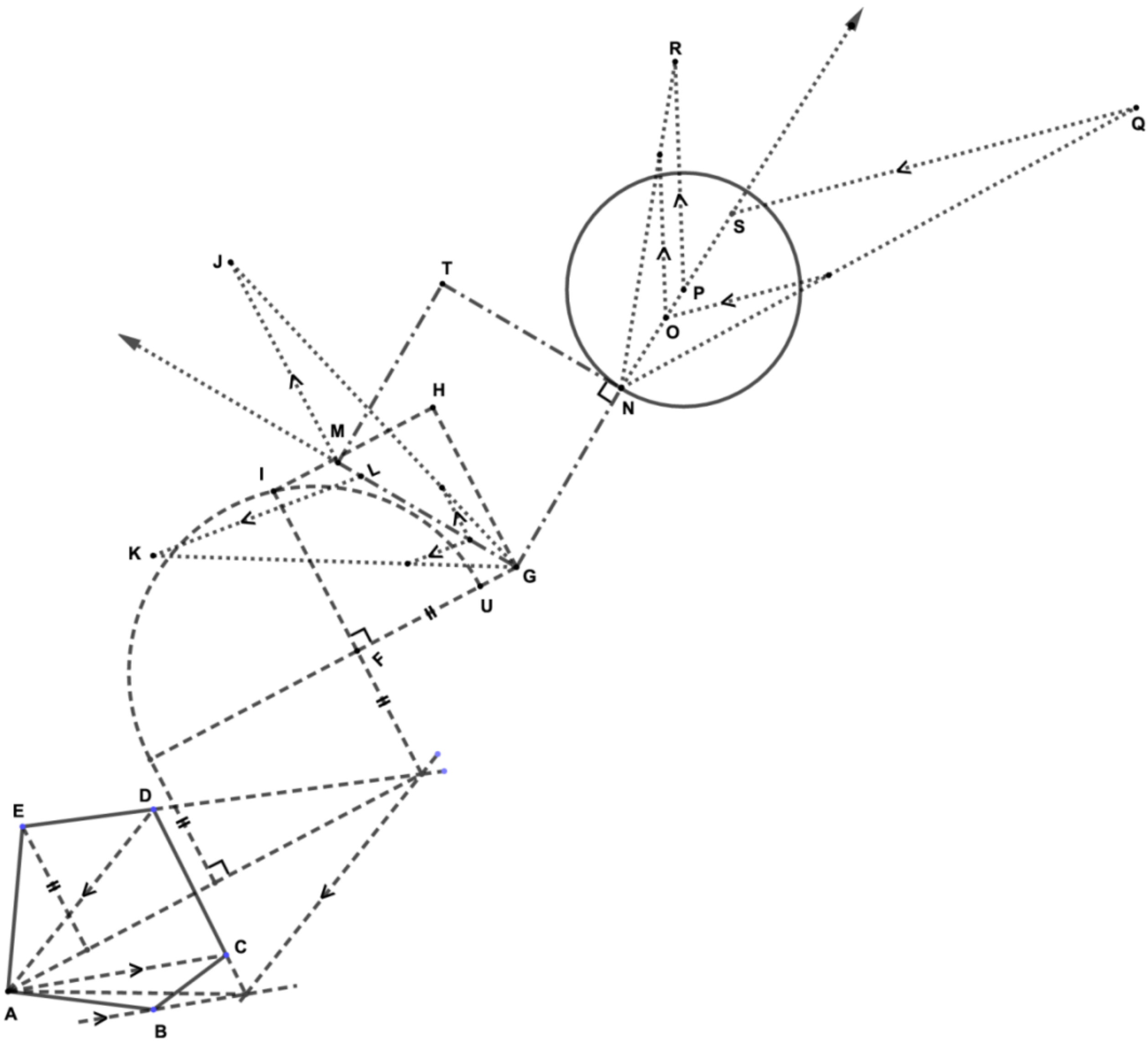

Figure 8

Proof ; Polygon ABCDE = Square FGHI

By construction, GJ = $\sqrt{123}$, GK = $\sqrt{100}$, GL = GF

$$\frac{GJ^2}{GK^2} = \frac{GM^2}{GL^2}$$

$$\frac{123}{100} \times GL^2 = GM^2$$

$$\text{ie., } GM^2 = 1.23 \times GL^2$$

$$= 1.23 \times GF^2$$

$$\text{Ar (Square GMTN)} = 1.23 \times \text{Ar (Square GFIM)}$$

$$= 1.23 \times \text{Ar (Polygon ABCDE)}$$

Again, by construction, NQ = $\sqrt{103993}$, NR = $\sqrt{33102}$, NS = GM

$$\frac{NQ^2}{NR^2} = \frac{NS^2}{NP^2}$$

$$\frac{103993}{33102} \times NP^2 = NS^2$$

$$\text{ie. } \pi r^2 = GM^2$$

$$= 1.23 \cdot GF^2$$

$$= 1.23 \cdot \text{Ar (Polygon ABCDE)}$$

∴ Area of Circle = 1.23 times area of the Polygon ABCDE

Conclusion

1. It is not difficult to convert a circle into square or square into a circle with error minimised as per one's wish.
2. Also, this single fixed method helps to construct a circle with area equal to any rational number of times of enlarging or shrinking the area of any *n*-sided convex polygon.

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The Bible.