TOTALLY SELF-RELIANT TRIGONOMETRY

A WAY TO CALCULATE ANY TRIGONOMETRIC FUNCTION OF 24 KEY ANGLES WITHOUT NEEDING A CALCULATOR

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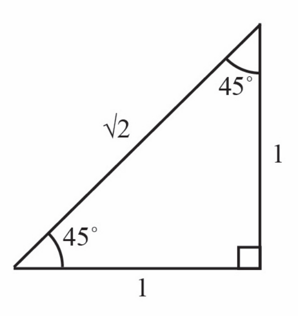
Before I begin, to elaborate a little further on the sub-title, I will show you how to get even the decimal equivalents of these trig functions for the said angle.

I like to play around with Vedic algorithms, because after awhile, I get deeper insights into the actual mathematics, which help me discover hitherto unknown tools. For this paper, I found that after much experimentation with adding and subtracting Vedic triples, I came up with even faster and more efficient methods to calculate any of the 6 trigonometric functions, without even needing triples operations. I was able to do this for 24 angles, each 15 degrees apart, from $0^o/360^o$ to 345^o.

We need to see this field of knowledge as 4 separate angle categories, each with its own special algorithms.

Category I: The "Mandatory Knowledge" (M) Triples, i.e. the 45^o right triangle, and the 30^o-60^o-right triangle

The 1-1-√2 triple for the 45^o right triangle is based on the fact that if one acute angle is 45^o, then so must be the other, since they are complementary to each other. This means the sides opposite these angles are congruent, since congruent base angles of any triangle generate equal opposite sides. This makes the right triangle here an isosceles one. So since the 2 legs are indeed congruent, we may as well give each leg the simplest number, 1. So with the Pythagorean Theorem, the hypotenuse must be √2.



In the math world, some reinterpret this into the unit circle schema, where the hypotenuse is 1. So to keep the hypotenuse as a unit value, the sides must be divided by $\sqrt{2}$, making each leg $1/\sqrt{2}$. Because of the distaste by some for radicals in the denominator, each leg is multiplied by $\sqrt{2}/\sqrt{2}$, resulting in unit circle sides $\sqrt{2}/2$.

So you might at times see the triple as, $\sqrt{2}/2$ $\sqrt{2}/2$ 1, which gives you an instantaneous cosine and sine

To better memorize the 30-60 right triangle triple, you could just remember that the height, or opposite side to the 30^o angle is always ½ the hypotenuse.

Hence $y = 1$, $r = 2$. i.e. 30^o) --- 1 2   Then we use $a^2 = c^2 – b^2$ to solve the base x.

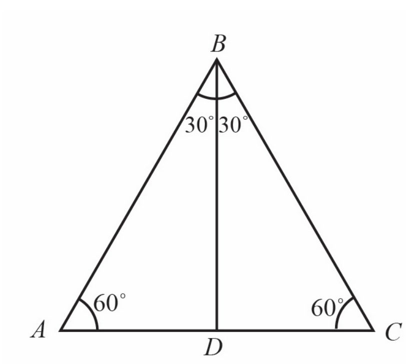
i.e. $a^2 = 2^2 – 1^2 = 3$, then get the square root. So the first element of the triple is $\sqrt{3}$.

So we have two triples: 30^o as $\sqrt{3}$ 1 2 and 60^o as 1 $\sqrt{3}$ 2.

One remaining mystery: Why is sin 30^o *exactly* ½? For those old enough to remember; before there were calculators, we had those trig tables, with "sine", "cosine", and "tangent", heading the columns, and every degree from 1 to 90, preceding the rows. Remember how each function with its degree had a mysterious looking 4-digit decimal approximation, each one with 4 complicated looking digits; all except for two: sin 30^o and cos 60^o each = exactly .5000. Why were these two so easy and convenient, where all the other angle trig values were so abstruse-looking?

We can derive the sin 30^o = ½ thusly:

Take an equilateral triangle (i.e. each vertex = 60^o). Draw an angle bisector to an opposite base to form two 30^o angles:



ΔABD ≅ ΔBCD   How do we know?   The SAS (side-angle-side proof)

Side BD is shared by both triangles , i.e. the reflexive property. That is the first S.

Angle ABD ≅ angle CBD: definition of angle bisector. That is the A.

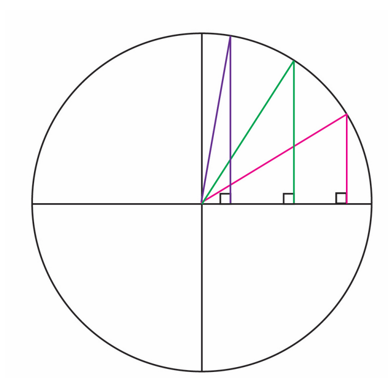
Side AB ≅ side BC: definition of equilateral Δ. That's the S on the other side of A.

Since both these triangles are congruent, then their corresponding sides AD and DC must equal each other. But that means two equal parts within side AC, the other side of this equilateral triangle. Two equal parts means halves. So we see AD is half of AB, and DC is half of BC. Also angles ADB and CDB are right angles. We know this because since both triangles are congruent, and these adjacent corresponding angles make up a straight line segment, i.e. 180^o, each must by 90^o. Thus, within each right triangle, we have a 30^o angle causing its opposite side (height in VM) to be half of its respective hypotenuse.

Category 2: The Quadrantal Angle (Q) Triples

I am assuming most readers know the Vedic triples for these angles. However, there is a good reason to go over them again. There is a certain pattern to them that will make it easier to master my last category, the 15^o offset angles.

Note that the quadrantal angles are the only angle type where the right triangle disappears and collapses unto itself as a single line segment. Ex. As you go from 0^o to 90^o, imagine a shifting right triangle that gets higher and narrower as we go counter-clockwise.



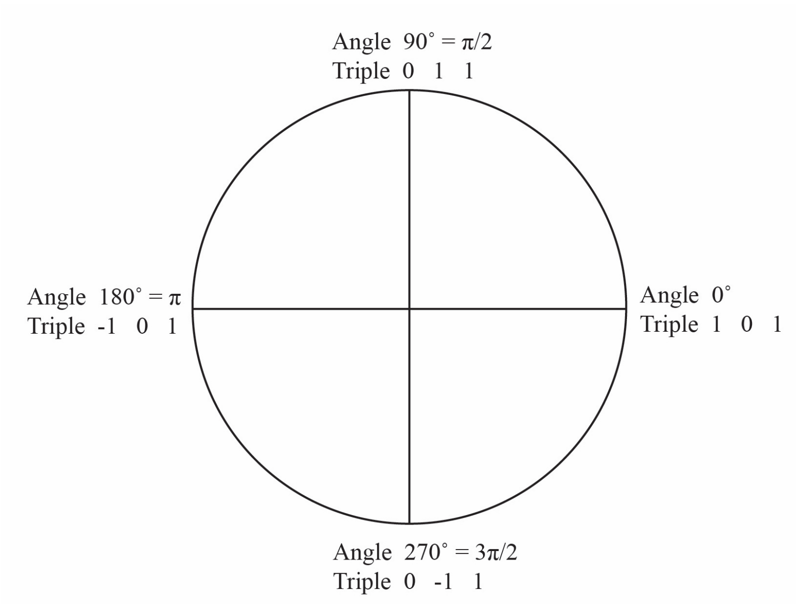
The vertical side or Vedic “height” is drawing closer and closer to the hypotenuse, as the latter itself converges onto the 90^o vertical ray. Exactly at 90^o, the vertical side, the hypotenuse, and the unit circle radius of 1 are all indistinguishable. It is now all vertical, with totally non-existent horizontal. Thus our triple for 90^o or $\pi/2$ radians. = 0 1 1.

Notice that whenever one straight leg element is 1, the other element is 0. In preparation for a later angle category, a strong vertical means a weak horizontal, and vica versa. Here at $\pi/2$ radians, we have a strong vertical and weak horizontal; actually a totally dominant vertical and non-existent horizontal. Without drawing the diagram, you can imagine why the Vedic triple for π radians or 180^o must be -1 0 1. We have a totally dominant horizontal for element 1 (going left, hence negative), and a non-existent vertical or 0 for element 2.

Indeed, even if you forget the memorization of the 4 quadrantal triples, just ask the following:

a) Is it on the *x*-axis (horizontal dominant) or the *y*-axis (vertical dominant)? So you know if it’s the former *x* case, then the Vedic base must be 1 (absolute value for now) and the Vedic height must be 0. If the latter *y* case, then the Vedic height is | 1 | and the base is 0.

b) Now for the sign of 1: if horizontal right ($0^o/360^o$) or vertical up (90^o), it’s + for base or height respectively. If horizontal left (180^o) or vertical down 270^o), it’s – for base or height respectively.



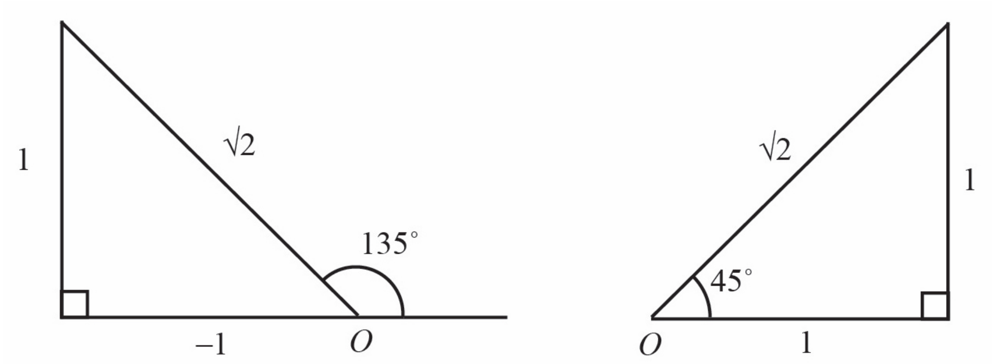
Category 3: The RQ – reference angle and quadrant method

This is explained more fully in my video 1 of the VM conference March 17-18, 2018. However, I can describe it briefly here.

There are 9 angles beyond quadrant 1 whose triples don't need Vedic addition or subtraction to solve. They each are either 30°, 45° or 60° away from the *x*-axis, the basis for calculating reference angles. We just make the triple for the reference angle, then adjust the sign(s) for the triple's *x* and *y* based on the angle's quadrant, and you've got the triple.

Example 1 135° (or 3π/4 radians). Reference angle = 45° (or π/4). So our base triple is 1 1 √2

Since 135° is in quadrant 2, we know the base (*x* value) is negative, and the height (*y* value) is positive. So the final triple for 135° = - 1 1 √2. And you can now figure your trig functions accordingly, for example, tan 3π/4(135°) = y/x = 1/-1 = -1.

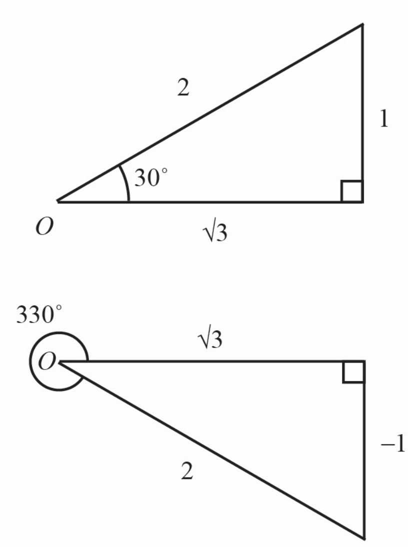


Example 2 330° (11π/6).

Reference angle = 30° so base triple is √3 1 2

Quadrant 4 means + for *x* and – for *y*
so now it's √3 - 1 2

Example 3 sin 330° is y/r or simply -1/2



Category 4: The 15^{o} Offset Method

There are eight angles among the 24 angles that Vedic triples can address that have a special character. These are the ones that are 15^{o} off from each of the 4 quadrantal angles as follows:

345^{o} and 15^{o}, offset from 0^{o}/360^{o}

75^{o} and 105^{o}, offset from 90^{o}

165^{o} and 195^{o}, offset from 180^{o}

255^{o} and 285^{o}, offset from 270^{o}

The purpose of this section is to show how to use deductive reasoning to find each triple without the need for any Vedic addition, subtraction, or half angle operations once one basic triple is established for 15˚.

The triple for 15˚ can be found by subtracting the triple for 30˚ from the triple for 45˚

45˚	1	1	√2
30˚	√3	1	2
15˚	√3+1	√3−1	2√2

Every single one of these angles has an identical triple re: the number pair for each element. They differ solely by the sign preceding each number of the pair.

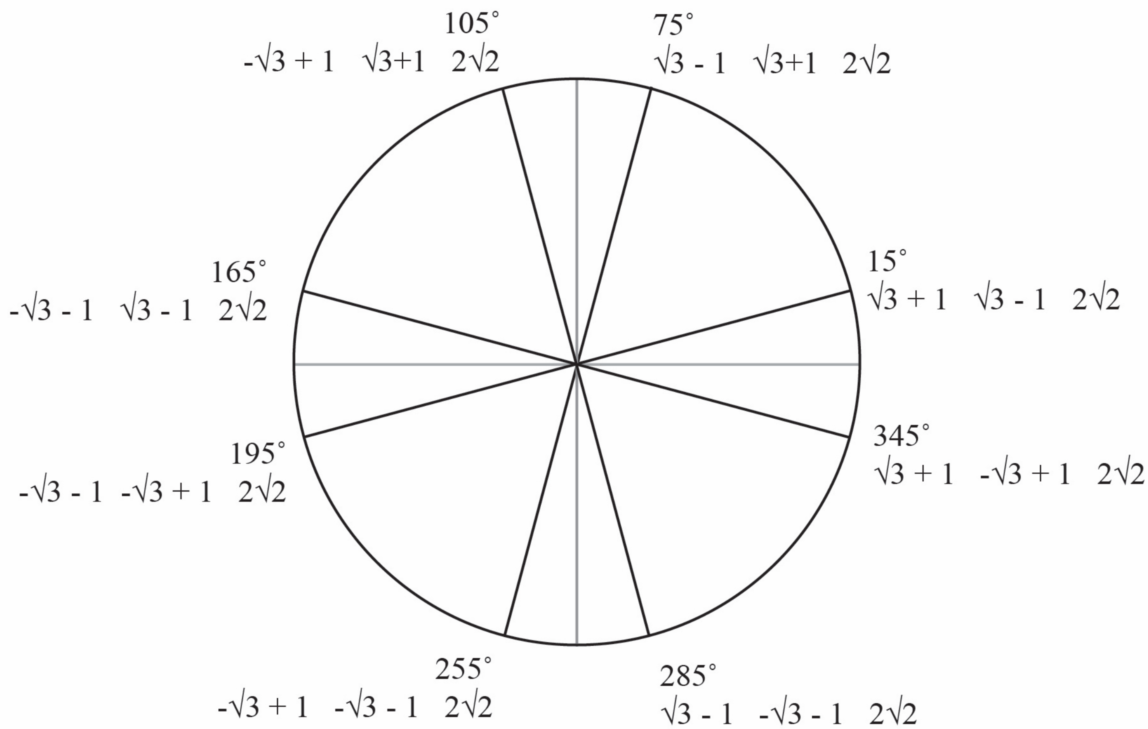
The basic template for all of them is:

+/-√3 +/-1	+/-√3 +/-1	2√2
base	height	hypotenuse

Here are all 8 of them with their own triples.

15^{o}:	√3 + 1	√3 – 1	2√2	195^{o}:	-√3 – 1	-√3 + 1	2√2
75^{o}:	√3 – 1	√3 + 1	“	255^{o}:	-√3 + 1	-√3 – 1	“
105^{o}:	-√3 + 1	√3 + 1	“	285^{o}:	√3 – 1	-√3 – 1	“
165^{o}:	-√3 – 1	√3 – 1	“	345^{o}:	√3 + 1	-√3 + 1	“

These are all shown on the diagram below.



Finding patterns

The quadrant of the angle tells us the sign layout for the √3 term of the base and the height. The √3 of the base mirrors the sign of x for that quadrant, and the √3 of the height mirrors the sign of y for that quadrant.

Thus:

Q1: from (+, +) gives us (√3…, √3…)

Q2: from (-, +) gives us (-√3…, √3…)

Q3: from (-, -) gives us (-√3 …,-√3…)

Q4: from (+, -) gives us (√3… , -√3…)

Check the above angles to confirm.

Now for the sign of the 1 following the $\sqrt{3}$:

Some angles are "strong horizontals". Those are the ones offset from the *x*-axis. i.e. 345^o, 15^o, 165^o, 195^o For the **base**, a strong horizontal going to the right has a +1, and going to the left has a – 1.

For example, 165^o is a strong negative horizontal, so its base has a – 1. And being in Q2, its full base is $-\sqrt{3} - 1$.

The other 4 angles are "strong verticals", i.e. offset from the *y*-axis,

i.e. 75^o, 105^o, 255^o, 285^o For the **height**, a strong vertical going up gives us a + 1, and going down, a -1.

Example 4 285^o is a strong downward vertical, so its height has a -1. And being in Q4, the full height is $-\sqrt{3} - 1$.

A strong horizontal always has a weak vertical (it climbs only slightly up or down), and a strong vertical always has a weak horizontal (it shifts only slightly to the right or left). This means a strong base has a weak height, and a strong height has a weak base. The implications for the sign preceding the 1: it's always opposite the intended direction. A weak base trying to go right will have a -1, or trying to go left will have a + 1.

Example 5 255^o, being a strong vertical, has a weak horizontal. It tries to go left, so its 1 is positive, the opposite of the intended direction.

So the base for 255^o is $-\sqrt{3} + 1$ (Q3 for the $\sqrt{3}$ term)

Its full triple is	$-\sqrt{3} + 1$	$-\sqrt{3} - 1$	$2\sqrt{2}$
	Weak (-) base	Strong (-) height	

Example 6 345^o, being a strong right horizontal, has a weak downward vertical. The height tries to go down, so it's a + 1, the opposite of the intended direction.

Being Q4, its height is $-\sqrt{3} + 1$.

The full triple is $\sqrt{3} + 1$ $-\sqrt{3} + 1$ $2\sqrt{2}$

Strong (+) base Weak (-) height

(rightward) (downward)

Now let's put this all together for some examples:

Example 7 105^o: Q2, strong upward vertical, weak leftward horizontal

So, being Q2, the √3s are $-\sqrt{3}$… $\sqrt{3}$…

Strong upward vertical means height = $\sqrt{3} + 1$, This means the weak leftward base is $-\sqrt{3} + 1$

Full triple for 105^o: $-\sqrt{3} + 1$ $\sqrt{3} + 1$ -

Example 8 195^o: Q3, strong leftward horizontal, weak downward vertical

The Q3 gives us to start: $-\sqrt{3}$… $-\sqrt{3}$…

The strong leftward base gives us a -1 and the weak downward height gives us a +1.

Triple for 195^o: $-\sqrt{3} - 1$ $-\sqrt{3} + 1$ -

With practice, this 15^o offset method will become more and more natural and rapid.

We are now ready to show the entire table of 24 Vedic triple angles, where each one can be calculated (or memorized) without the need for any Vedic triple operation. (This is *not* to be interpreted as an effort to do away with Vedic triple addition, subtraction, double or half angle algorithms because, without them, you cannot get to this next level. I only came upon my reference angle, quadrant method and the current 15^o offset method after *extensive* experience with Vedic triple operations. We might call this entire enterprise *guided self-discovery*, a term coined by a great American math educator - Marilyn Burns, who believes when students with good fundamentals are guided by a skillful teacher, they can discover "on their own" new math insights.)

Here is the legend:

M: mandatory knowledge, like the triple for 45^o

Q; quadrantal angles, also mandatory (but could be approached from common sense)

RQ: my reference angle – quadrant method

15 OS: the 15^o offset method

Degrees	Method
15	15 OS
30	M
45	M
60	M
75	15 OS
90	Q
105	15 OS
120	RQ
135	RQ
150	RQ
165	15 OS
180	Q

Degrees	Method
195	15 OS
210	RQ
225	RQ
240	RQ
255	15 OS
270	Q
285	15 OS
300	RQ
315	RQ
330	RQ
345	15 OS
360/0	Q

To even better memorize these patterns, there is a symmetry effect between each quadrantal angle inclusive: Q, 15 OS, RQ, RQ, RQ, 15 OS, Q (though in quadrant 1, we replace the RQ with M: i.e. Q, 15 OS, M, M, M, 15 OS, Q).

Final Step – Manually calculating the decimal equivalent of a trig function's value.

So we finally have our trig function of our target angle. You will notice how in most cases, we have a mixture of square roots and integers. What I find irksome about this is: what does an expression like that mean to a rational person?

For example, would we be able to think of a practical meaningful use for $\frac{\sqrt{6}+\sqrt{2}}{4}$?

Wouldn't it be more convenient to represent this value as an actual single number?

I've come up with a simple, practical method to do just this – and never even use a calculator to help us. I notice that in all the 24 angles and their trig functions, only 3 radicals appear:

$\sqrt{2}$, $\sqrt{3}$, or $\sqrt{6}$. If we can memorize their mixed decimal value, we can basically calculate any permutation of them with integers. I choose 3 significant digits, i.e. the integer before the decimal and the two after.

So: $\sqrt{2} \sim 1.41$, $\sqrt{3} \sim 1.73$, $\sqrt{6} \sim 2.45$

For good measure, we could also use: $\sqrt{3}/2 \sim 0.866$, i.e. sin 60° and cos 30° and $\sqrt{2}/2 \sim 0.707$, i.e. sin and cos of 45°

Here, the 3 significant digits are all to the right of the decimal.

Example 9 cos 225° ($5\pi/4$)

The RQ method gives us reference angle 45° and quadrant 3: -1 -1 $\sqrt{2}$

Cosine of that: $-1/\sqrt{2}$ or better: $-\sqrt{2}/2$, which we memorized is -0.707.

Example 10 tan 195° We should recognize a 15° offset here:

(1) Q3: $-\sqrt{3}$ … $-\sqrt{3}$… -

(2) Strong horizontal left, so x element: $-\sqrt{3} - 1$

(3) Weak vertical down, so y element: $-\sqrt{3} + 1$

So tan 195° is y/x which gives, $\dfrac{-\sqrt{3}+1}{-\sqrt{3}-1} \times \dfrac{-\sqrt{3}+1}{-\sqrt{3}+1} = \dfrac{3-2\sqrt{3}+1}{2} = 2-\sqrt{3}$

2 – 1.73: bar number subtraction or just seeing it: 0.27 is our answer.

Example 11 csc 75° 15 OS, Q1 $\sqrt{3}$… $\sqrt{3}$…

Strong vertical up: so y element is $\sqrt{3} + 1$

Weak horizontal right so x element is √3 – 1 (with csc, x element not needed)

Vedic triple - √3 + 1 2√2

So, csc 75^o is r/y which gives,

$$\frac{2\sqrt{2}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{2\sqrt{6}-2\sqrt{2}}{2} = \sqrt{6}-\sqrt{2} = 2.45-1.41 = 1.04$$

Okay – we are now ready to put this all together:

Decide on Vedic triple method: M (mandatory), Q (quadrantal), RQ (reference angle quadrant method), or 15 OS (15^o offset method)

After getting the triple based on an above method, get the trig function value as an integer, or mix of square root(s) and integer.

If needed, calculate manually the decimal value.

We will do 8 mixed cases: (I will include radian equivalent of the M, Q and RQ angles)

1. tan of 270^o (3π/2) Q angle, Vedic triple is 0 -1 1 tan is y/x = -1/0 = undefined

(Actually you should know this from the tangent graph – i.e all odd π/2s are vertical asymptotes, i.e. undefined)

2. sec of 30^o (π/6) M angle √3 1 2, actually since sec is reciprocal of cosine, we

don't need *y*: √3 - 2 sec is 2/√3 or 2/1.73. We could use flag division.

But since $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$, the easier calculation is $2 \times 1.73 \div 3 = 3.46 \div 3 = 1.15$

3. csc 330^o (11π/6) RQ method ref angle 30^o and Q4 (+,-) - -1 2

This one is easy: r/y = -2

4. cot 285^o 15 OS method, Q4, strong vertical down, weak horizontal right

Q4: √3 … -√3 … -

Vertical added: √3 … -√3 – 1 -

Horizontal added: $\sqrt{3}$ - 1 -$\sqrt{3}$ – 1 -

So, cot is x/y which gives,

$$\frac{\sqrt{3}-1}{-\sqrt{3}-1} \times \frac{-\sqrt{3}+1}{-\sqrt{3}+1} = \frac{-3+2\sqrt{3}-1}{2} = \sqrt{3}-2 = 1.73-2 = -0.27$$

5. cos π Q angle of 180° -1 0 1 easy: x/r = -1/1 = -1

6. sin 240° (4π/3) RQ situation ref angle of 60° and Q3 gives us:

-1 -$\sqrt{3}$ 2 y/r = -$\sqrt{3}$/2

We should have as memorized -0.866 or use simple short division.

7. sec 165° 15 OS case Q2: -$\sqrt{3}$… $\sqrt{3}$…

strong horizontal left: : -$\sqrt{3}$ -1 - 2$\sqrt{2}$ (We don't need *y*)

Sec 165° is r/x $= \frac{2\sqrt{2}}{-\sqrt{3}-1} \times \frac{-\sqrt{3}+1}{-\sqrt{3}+1} = \frac{-2\sqrt{6}+2\sqrt{2}}{2} = -\sqrt{6}+\sqrt{2} = -2.45+1.41 = -1.04$

8. tan 5π/6 or 150° RQ method 30° at Q2 -$\sqrt{3}$ 1 2

$$\text{tan is } y/x = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} = -\frac{1.73}{3} = -0.576$$

These decimal and integer answers are much more meaningful than answers with square roots, and can more easily be used to solve real-life applications.

Conclusion

This approach, using symmetry to finding the triples and exact trigonometric values of 24 angles, extends student's knowledge of trigonometry and unravels difficulties often experienced with angles beyond the first quadrant. The fact that these calculations can all be done without resort to calculators will be of considerable benefit to the millions of students, such as those in India and China and elsewhere in Asia where calculators are not used until university.