SUM AND DIFFERENCE OF SQUARES

Kenneth Williams, Vedic Mathematics Academy, U.K.

Abstract

Chapter 31 of Sri Bharati Krishna Tirthaji's book "Vedic Mathematics"[1] is titled "Sum and Difference of Squares". The chapter covers just four pages and though the first section on "Difference of Two Square Numbers" is very clear, the second part on "The Sum of Two Square Numbers" raises a number of questions as it does not appear to answer the challenge it sets itself.

This paper will explain these discrepancies, suggest what may be missing from the second part of the chapter and why the particular triples used may have been chosen. An interesting extension of the first part of the chapter will also be described.

1. Introduction

Given the brief and apparently truncated nature of the second part of Chapter 31, as well as Tirthaji's curious choice of certain Pythagorean triples (hereafter simply referred to as 'triples') with large integral values, a closer analysis of this chapter would be of some value.

This paper is in three parts.

Section 2: A corollary is given to the first part of the chapter showing how to find two sides of a right-angled triangle given one (integral) side.

Section 3: Given that the triples generated in the second part of the chapter have quite large integer values, the question is addressed as to whether there is some significance in the triples chosen since many, more manageable, triples could have been used.

Section 4: Discusses why it is that Tirthaji appears not to have solved the problem set: to express a given number as the sum of two square numbers.

We may define a triple as three numbers p, q, r, all positive integers such that $p^2 + q^2 = r^2$.

2. Difference of Two Square Numbers

2.1 Here Tirthaji uses the identity $ab \equiv \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$, $a > b$ to express any given number as a difference of squares.

The given number, N, is written as a product ab and the values of $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are found.

So, given $N = 12$ we can have $12 = 12\times1$ or 6×2 or 4×3.
For $12 = 6\times2$ for example we have $a = 6$, $b = 2$ and so $12 = 4^2 - 2^2$.

And since a and b can take fractional values we have an infinite number of ways of expressing any given integer as a difference of two square numbers.

2.2 A Corollary

Taking $12 = 4^2 – 2^2$ we may note that if we were dealing with right-angled triangles, this would indicate one with sides $\sqrt{12}$, 2 and 4, the 3rd side being the hypotenuse.

If we wish to generate a triple, where the three sides of the right-angled triangle are *all positive integers* the above process would not work. It can however be adapted to work.

2.2.1 Example 1

Suppose we wish to find a triple with a given side N, say $N = 5$.

The process described in section 2.1 above would mean $a = 5$ and $b = 1$ and would lead to $5 = 3^2 – 2^2$. This would not give us a triple as defined in the introduction since we get $\sqrt{5}$, 2 and 3.

However if we start with $N^2 = 25$ we can say $25 = 25\times1 = 13^2 – 12^2$.

And this means our triangle would have sides 5, 12, 13, i.e. a triple with one side having a given value.

2.2.2 Example 2

Similarly, if we want a triple having a side $N = 9$ we write 81 as a product.

For $81 = 81\times1$ we get $9^2 = 41^2 – 40^2$.
For $81 = 27\times3$ we get $9^2 = 15^2 – 12^2$.

So in this case we have two solutions: the triples 9, 40, 41 and 9, 12, 15.

2.2.3 Same parity required in the chosen product.

It is worth noting that the two parts of the product formed from N^2 must be either both even or both odd. This is because if they are of opposite parity then the values of $\frac{a+b}{2}$ and $\frac{a-b}{2}$ would be fractional.

For example, if $N = 4$ then 8×2 could be used (this gives 4,3,5), but not 16×1 (which gives 4, 7½,8½, which is not a triple).

2.2.4 A triple can always be found.

We may show that, provided $N^2 > 4$, a triple with a given side can always be found.

If N^2 is odd the product used can be written $N^2 \times 1$ (the product of two odd numbers).
If N^2 is even it must have a factor of 4 and so can be expressed as the product of two even numbers. Hence a triple can always be found containing a given side.

3. Tirthaji's Chosen Triples

3.1 Generating the Triples

Tirthaji shows two methods to arrive at the illustrative triples in the second section of the chapter.

In the first method he uses the following examples:

$$9^2 + 402 = 412,$$
$$352 + 6122 = 6132,$$
$$572 + 16242 = 16252,$$
$$1412 + 99402 = 9941^2.$$

In the second method he uses:

$$7^2 + 24^2 = 25^2,$$
$$9^2 + 40^2 = 41^2,$$
$$35^2 + 612^2 = 613^2,$$
$$57^2 + 1624^2 = 1625^2,$$
$$141^2 + 9940^2 = 9941^2.$$

So four triples are repeated.

We must note the large numbers involved, which could easily have been avoided.

Could there some significance in the choice of triples? Could Tirthaji have been trying to indicate something without being explicit?

In both cases Tirthaji has a starting value he labels a, which is used to generate the triple and which turns out to be the first number in the triple.

So for $9^2 + 40^2 = 41^2$ he starts with $a = 9$ and generates the triple from there.

3.2 Values of the Small Angles.

Now suppose we focus on the last three triples, the ones with the large values:

$$35^2 + 612^2 = 613^2,$$
$$57^2 + 1624^2 = 1625^2,$$
$$141^2 + 9940^2 = 9941^2.$$

Because the two longest sides are large and differ by only one unit the triangle will contain a small angle. Let us note those angles, A, in radians:
352 + 6122 = 6132, A = 0.057 to 3 decimal places,
572 + 16242 = 16252, A = 0.035 to 3 decimal places,
1412 + 99402 = 99412, A = 0.0141 to 4 decimal places.

These approximate angles show surprising relationships:
The triple generated from 35 has angle 0.057,

The triple generated from 57 has angle 0.035,
The triple generated from 141 has angle 0.0141,

We see that the triples generated from 35 and 57 are complementary, and the one generated from 141 is self-complementary.

This would suggest that Tirthaji may have been hinting at a relationship between the generating value a and the angle in the generated triple.

3.3 Connection Between *a* and *A*.

In fact closer study[2] reveals that the relationship being indicated is that the product of a and A is approximately 2. That is, for large values of a (or small values of A), $aA \approx 2$.

For the three triples Tirthaji mentions we find:

$$35 \times 0.057 = 1.995,$$
$$57 \times 0.035 = 1.995,$$
$$141 \times 0.0141 = 1.9881.$$

This approximation becomes more accurate for smaller angles and makes it quite easy to find the angle in a given triple if that angle is small, or to generate a triple containing a given small angle.

3.4 Finding an Angle in a Given Triple

Example: given $a = 101$, generate the triple and find its angle.
Using Tirthaji's method we generate 101, 5100, 5101.
And the angle will be $A = 2 \div 101 = 0.019802$ radians.

The actual angle, to 6 decimal places, is 0.019801, differing only by 1 in the last place.

3.5 Finding a Triple with a Given Angle

Example: Find a triple with an angle of 0.06 radians.
We find $2 \div 0.06 = 33$ (to 2 significant figures) $= a$.
The triple then generated from this 33 is 33, 544, 545 .

This chapter led to my own development of an arithmetic of triples[2] back in 1981which turned out to have a great many applications, leading to many mathematical methods which were hitherto unknown and showing a remarkable unity, and alliance with the Sutras of Vedic mathematics.

For example through a carefully chosen definition of triple addition under the Vertically and Crosswise Sutra we can find trig ratios of compound angles, solve trig equations of various types, find the distance of a point from a line and other applications in coordinate geometry, perform transformations, square root of a complex number in one line, etc. There are applications in applied mathematics and a 3-dimensional version of triples has many applications too.

4 Expressing a Number as the Sum of Two Squares

4.1 Tirthaji's Text and the Question Arising

Having completed the first part of the chapter on expressing a given number as the difference of two squares the next section is headed "The Sum of Two Square Numbers" and proceeds as follows:

"Inasmuch, however, as $a^2 + b^2$ has no such corresponding advantage or facilities etc., to offer, the problem of expressing any number as the sum of two square numbers is a tough one and needs very careful attention. This, therefore, we now proceed to deal with."

We expect therefore to find out, from a given number, how it may be expressed as the sum of two squares (though we must expect that this may not be always possible, for example 11 cannot be expressed as a sum of squares).

But all Tirthaji does is show us how to generate a triple from a given starting value, *a*.

So, for example:

"If $a = 9$, its square (81) = 40 + 41 $\therefore$ $9^2 + 40^2 = 41^2$".

The closest we can get to the expressing of a number as the sum of two squares from this is that $41^2 = 9^2 + 40^2$, i.e. that $1681 = 9^2 + 40^2$.

But this is not what we were expecting: we were expecting to express *the given number* $a = 9$ as a sum of squares.

This is practically the end of the chapter, the only remaining part is on how to handle the case where an even number is given, which bears only on how to generate the triple when starting with an even number.

What then is the explanation for this? Is there a section missing – could some text have been lost, or perhaps, since Tirthaji was dictating to an amanuensis, the intention to complete chapter 31 before going on to the next chapter was forgotten? There are other possible explanations too.

The following is an attempt to complete Chapter 31 by showing how the earlier section could lead up to a way of expressing a number as the sum of two squares.

4.2 A Possible Answer to the Question

In his second method for generating Triples Tirthaji writes:

"... the square of the given number is the sum of two consecutive integers at the exact middle. For instance, if 7 be the given number, its square = 49 which can be split up into the two consecutive integers 24 and 25.

$\therefore\ 7^2 + 24^2 = 25^2$. Similarly,

(1) If $a = 9$, its square (81) = 40 + 41 $\therefore\ 9^2 + 40^2 = 41^2$"

Three more, similar, examples follow.

We may note from this that the largest of the three numbers, M say, in the triple can be obtained from the given number a by means of the formula $M = \frac{a^2+1}{2}$.

For example, if $a = 9$, $M = \frac{9^2+1}{2} = 41$.

So *in reverse* we may write a in terms of M: $a = \sqrt{(2M-1)}$.

Example Suppose we wish to express 41 as the sum of two squares, then $M = 41$ and $a = \sqrt{(2M-1)} = 9$.

Now if we apply Tirthaji's idea from the above quote, we find two consecutive integers adding to 9.

We get 4 and 5 and so $41 = 4^2 + 5^2$.

That is we have expressed the given number, 41, as the sum of two squares.

So the method is:
1) Take the given number, M, and find $a = \sqrt{(2M-1)}$,
2) split a into two consecutive integers, b and c say, and write $M = b^2 + c^2$.

4.3 Further Examples

Similarly if we wish to express 25 as the sum of two squares, $M = 25$ and $a = \sqrt{(2M-1)} = 7$.

Two consecutive integers adding to 7 must be 3 and 4, so $25 = 3^2 + 4^2$.

And for $M = 13$, $a = 5 = 2 + 3$. So $13 = 2^2 + 3^2$.

4.4 Proof

This result can be proved as follows.

If $\sqrt{(2M-1)}$ is split into two consecutive integers they will be $\frac{\sqrt{2M-1}-1}{2}$ and $\frac{\sqrt{2M-1}+1}{2}$.

We therefore expect that the sum of the squares of these two expressions will be M and this is easily shown to be true.

4.5 The Full Solution

This method can only be applied for values of M for which $(2M - 1)$ is a perfect square. So it is not a complete answer to the question of expressing a number as the sum of two squares. It would not work for example with $a = 20$ (i.e. 2^2+4^2).

However the method shown above (Sections 4.2, 4.3) would follow on nicely from the second part of Tirthaji's chapter and answer the question of expressing a number as the sum of two squares for the numbers that are the largest elements of the triples he generates.

Tirthaji concludes chapter 31 with:

"There are many other simple and easy methods by which we can tackle the problem (of $a^2 + b^2 = c^2$) by means of clues and conclusions deducible from $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $8^2 + 15^2 = 17^2$ etc. But into details of these and other allied matters we do not now enter."

This is very interesting because the first two of the three triples quoted are ones that appear in the chapter itself, but the third one ($8^2 + 15^2 = 17^2$) is of a different class (as 15 and 17 differ by 2 units and not 1).

This may be a hint that this method is not intended to be the complete answer to the question, and also perhaps a hint how to develop this approach further.

5. Concluding Remarks

Tirthaji's method for expressing a number as the difference of two squares is clear and complete. Maybe Tirthaji thought his examples on expressing a number as the sum of two squares were enough to give the clue how to complete the expected method.

We may note that no mention is made in the chapter of the Sutras. The chapter as a whole seems to indicate methods that are required for other work, and in fact Tirthaji opens the chapter with:

"Not only with regard to questions arising in connection with and arising out of Pythagoras' Theorem (which we shall shortly be taking up) but also in respect of matters relating to the three fundamental Trigonometrical-Ratio-relationships (as indicated by the three formulae . . .) etc., etc. we have often to deal with the difference of two square numbers, the addition of two square numbers etc., etc. And it is desirable to have the assistance of rules governing this subject and benefit by them.

So it would appear that there are ways in which we need to use the techniques described in further work, not specified in detail by Tirthaji. This is therefore another area for research.

One additional area would be in coordinate geometry where we go into higher dimensions. For example, given the triple 9,12,15 we may express 15^2 as a difference of two squares and obtain the ‘quadruple’ 9,12,20,25 (quadruple means: $9^2+12^2+20^2=25^2$).

Or given the triple 5,12,13 we may express 5^2 as a sum of two squares (by finding $\sqrt{(2\times5^2 - 1)} = 7$) and obtain the quadruple 3,4,12,13. These processes are also reversible so that we can also go from a higher order ‘tuple’ to a lower order ‘tuple’.

The full solution for expressing any given positive integer (capable of expression as a sum of squares) as a sum of two squares will be the subject of a later paper.

References

[1] Bharati Krishna Tirthaji Maharaja, (1965). Vedic Mathematics. Delhi: Motilal Banarasidas,.

[2] Williams. K. R. (2013). Triples. U.K.: Inspiration Books.