BUILDING POLYNOMIALS FROM SEPARATE ROOTS:

AN ULTIMATE VINDICATION OF VERTICALLY AND CROSSWISE

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One of the more challenging issues in teaching a pre-calculus course is showing how to get around the clumsy lengthy process of multiplying (x – root) factors to build a higher degree polynomial. It gets really involved when multi-lying two trinomials together. Vertically and crosswise (V & C) multiplication changes this process to a trivial one.

I will present this topic in 3 progressive stages:

- Western multiplication – crude method
- Western multiplication – a more streamlined approach
- Vertically and crosswise – easily the best approach.

Case 1: A Prototype Example of Multiplying Separate Roots

(I am deliberately keeping things simple here: there will be no imaginary or complex roots – just integer roots, and the leading coefficient of the highest degree term will just be 1.)

Suppose you are given the roots (also could be called "zeros" or "*x*-intercepts") as -5, -1, 3, and 4. This means you are building a 4^{th} degree polynomial since your highest term will be to the fourth power.

So you must convert the roots to factors, i.e. (x – root).

So are factors are $(x+5)$, $(x+1)$, $(x-3)$ and $(x-4)$.

You would normally multiply two factors to get quadratic A, then the other two factors to get binomial B, then finally multiply the two quadratics together.

So let's say we pair up $(x+5)(x+1)$ and $(x-3)(x-4)$. Of course, you could pick any pairs you want.

The Crude Western Method

Inexperienced students would use FOIL, to obtain x^2+6x+5 and $x^2-7x+12$

More experienced students can do these metally.

The next step is to multiply the two quadratics. The Western approach is to multiply one quadratic by each term of the other and then collect up like term.

This produces $x^4+6x^3+5x^2$, $-7x^3-42x^2-35x$ and $12x^2+72x+60$

This is a mess, because now you must group together all the like terms, from highest exponent to lowest.

The final expression is $x^4 - x^3 - 25x^2 + 37x + 60$.

Western Method – More Streamlined

I thought this one up (before I knew about Vedic Math), but I'm sure any intelligent math teacher has come up with this also.

For the trinomials as factors, as you are doing your distributive multiplying, mentally keep track of the exponent in the partial product, then vertically align it under the identical exponent in the previous partial product.

Or simply with each new partial product, start one space to the right in the next line.

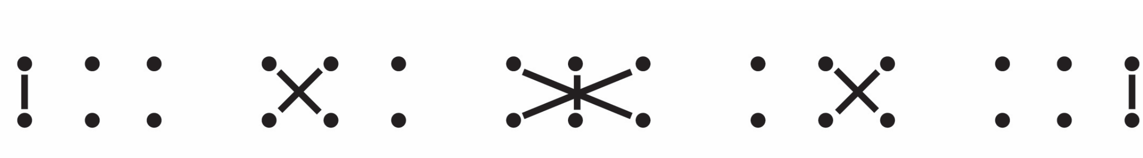
$$\begin{array}{rrrrr}
x^4 & +6x^3 & +5x^2 & & \\
 & -7x^3 & -42x^2 & -35x & \\
 & & 12x^2 & +72x & +60 \\
\hline
x^4 & -x^3 & -25x^2 & +37x & +60
\end{array}$$

Vertically and Crosswise

This is easily the best method. For multiplying binomials, the following pattern can be used to obtain the answer in one line:

$$\begin{array}{l}
x+5 \\
x+1 \\
\hline
x^2+6x+5
\end{array}
\quad \text{and} \quad
\begin{array}{l}
x-3 \\
x-4 \\
\hline
x^2-7x+12
\end{array}$$

So far, this is pretty straightforward. However, now we must multiply the 2 trinomials.



If you are new to this, refer to the above schema.

The steps are:

1. $x^2 \times x^2 = x^4$
2. $x^2 \times -7x \ + \ 6x \times x^2 \ = \ -x^3$
3. $12 \times x^2 \ + \ 6x \times -7x \ + \ 5 \times x^2 \ = \ -25x^2$
4. $6x \times 12 \ + \ 5 \times -7x \ = \ 37x$
5. $5 \times 12 = 60$

This can be set out in one line as,

$$\begin{array}{l} x^2 + 6x + 5 \\ \underline{x^2 - 7x + 12 \qquad\qquad} \\ x^4 - x^3 - 25x^2 + 37x + 60 \end{array}$$

You really can't appreciate the beauty of V & C until you do a lot of mental practice.

What I personally find gratifying about V & C is not so much the lightning speed, but even better, the way it automatically orders the exponents in consecutive descending order without you worrying about gathering together like terms from a scrambled hodgepodge.

Case 2: A pre-calculus situation of building a 4th degree polynomial from both integer and complex roots, and a > 1 exponent for the fourth degree term.

I won't even bother with the Western distributive method here. It would be embarrassingly cumbersome. Even the streamlined method could present problems. I am going straight to VM.

The problem: Given that 4, -5 and 3 + 2i are roots of a 4th degree polynomial $f(x) = 0$ and also that $f(2) = 140$, find the polynomial.

Since 3 + 2i is a root then so is the complex conjugate 3 – 2i.

Step 1: Multiplying pairs of factors

Pair off the integer roots, change to their factor versions, and multiply them.

Roots 4 and – 5 mean (x – 4)(x + 5). V & C immediately gets you $x^2 + x - 20$.

Pair off the complex roots

$\left\{x-(3+2i)\right\}\left\{x-(3-2i)\right\} \Rightarrow (x-3-2i)(x-3+2i)$

We are multiplying two trinomials using the same pattern as before.

$$\begin{array}{l} x-3-2i \\ \underline{x-3+2i} \\ \underline{x^2-6x+9-4i^2} \\ x^2-6x+13 \quad \text{since } i^2=-1 \end{array}$$

Step 2: Multiply the 2 trinomials

$$\begin{array}{l} x^2+x-20 \\ \underline{x^2-6x+13} \\ x^4-5x^3-13x^2+133x-260 \end{array}$$

Step 3: Dealing with $f(2)=140$

Evaluating the above at $x=2$ yields -70 and not 140. This tells you that the actual polynomial has a scale factor of -2. Each term must therefore be multiplied by -2 to give,

$-2x^4+10x^3+26x^2-266x+520$ which is the required polynomial.

Concluding Remarks

Vertically and crosswise gives a very neat and fast way for multiplying polynomials together in one line and should be fully utilised by students.