Algebraic Patterns associated with Vedic Mathematics to Shorten Integration of Quadratic Formulae

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Abstract

The Vedic Mathematics approach has proved effective in helping those preparing for pre-university professional examinations. Here we consider a particular kind of question in such examinations: multiple choice questions requiring students to select the correct answer for an integral with a quadratic denominator. This article shows how to shorten time spent on such questions by using principles associated with Vedic Mathematics. Classifying quadratics according to their first derivatives and discriminants gives an immediate classification of the kinds of result to be expected from the process of integration. The method allows immediate selection of the correct answer in multiple choice questions without entering into details of calculations. These particular kinds of question are among the most difficult in professional mathematics examinations, and can lead to the greatest loss of time, sapping candidate's confidence to the greatest extent. Providing simple and easily remembered methods to short cut to the correct solution enables candidates to complete this difficult examination section without frustration, or loss of confidence and corresponding loss of emotional energy.

Keywords: Mathematics; Vedic Mathematics; Integration; Quadratic equation

Introduction

Learning mathematics can be very challenging for some students. Difficulties in learning can decrease student confidence and compound the challenges for both students and teachers, who may find helping such students learn equally difficult. Such situations may be compounded when a student wants to take science or engineering courses at university, and obtaining good grades in mathematics may seem an impossible challenge, leading to Maths Anxiety (Maloney & Beilock 2012) and its associated problems. Any means to remedy these challenges are worth identifying and making known to the maths teaching profession. This article discusses a possible approach based on a particular form of pattern recognition, which has been found to decrease Maths Anxiety and improve high school (Submitted for publication), 12th Grade, final examination results.

Pattern recognition and representation play major roles in learning mathematics (Wittmann 2005; Steiner 2000; Resnik 1981). Such patterns may be numerical, algebraic or geometric in nature. Mathematics teaching that emphasizes patterns and their recognition can improve students' problem solving ability and make the process more rewarding for teachers. When patterns indicated by the teacher are recognised by students and reduce calculations by large amounts, students appreciate the method, and gain greater enjoyment from the processes of learning mathematics and working out the required examples.

Vedic Mathematics is a collection of formulae suggested by Sri Bharati Krishna Tirtha, Shankaracharya of Puri (Bharati Krsna Tirthaji Maharaja 1992), indicating simple looking patterns of calculation, which usually reduce cumbersome, lengthy calculations from five, six or more lines to one or two lines (Bhardwaj et al. 2012). Such methods of minimising length of calculation have even been used in industry, for example to design faster, smaller multiplier circuits, and a Fast Fourier Transform processor with improved calculation speed (Gupta et al. 2012). The approach may also be used effectively in competitive exams which limit students' time in arriving at accurate final answers. In such examinations, limitations of time may lead to students trying to increase calculation speed and consequently compromise their accuracy. Examples include engineering entrance exams and the preceding 10+2 pre-university board examinations.

Existing Vedic Maths method increases calculation speed while solving integrals with quadratic denominators that can be factorised. Vedic Maths and Conventional methods are compared here.

Example: $\int \frac{1}{x^2-5x+6} dx = \int \frac{1}{(x-2)(x-3)} dx = \int \frac{A}{(x-2)} dx + \int \frac{B}{(x-3)} dx$

Vedic maths suggests an application of the procedure, 'Transpose and Apply', as follows.

First let $x = 2$ (equating the factor $x - 2 = 0$) in LHS of the equation, giving $\frac{1}{2-3} = -1$ which is A.

Next let $x = 3$ (equating the factor $x - 3 = 0$) in LHS of the equation, giving $\frac{1}{3-2} = 1$ for B.

Finally integrate obtaining, $I = -\log(x-2) + \log(x-3) + C$. Note how using this procedure converts the solution to the problem into a one line mental calculation.

The laborious conventional method requires multiplying out the polynomial fraction:

$$\int \frac{1}{(x-2)(x-3)} = \int \frac{A}{(x-2)} + \int \frac{B}{(x-3)}$$

Consider $\frac{1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$

$$\Rightarrow \frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow 1 = A(x-3) + B(x-2)$$

$$\Rightarrow 1 = Ax - 3A + Bx - 2B$$

By equating coefficients of x and constant terms

$0 = A + B$ and $1 = -3A - 2B$ and solving them we get $A = -1 \; and \; B = 1$.

Now substitute values of A and B and finally integrate to get

$$I = -\log(x-2) + \log(x-3) + C.$$

Similarly, multiple choice questions involving integrals with quadratic denominators that cannot be factorised which appear in exam papers are especially challenging to students. They require the examinee to reduce the quadratic form to standard format by the process of completing the square of the quadratic function, after which the correct formula must be identified and applied. This is the usual method discussed in any integral calculus books (Ranganath 2013; Gradshteyn & Ryzhik 2007). The student is obliged to memorise the following standard integral formulae to obtain correct answers.

1) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} log \left|\frac{x-a}{x+a}\right| + c$ 2) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} log \left|\frac{a+x}{a-x}\right| + c$

3) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c$ 4) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2+a^2}\right| + \mathrm{c}$

5) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + c$ 6) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$

7) $\int \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right) + c$

8) $\int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} log\left|x + \sqrt{x^2-a^2}\right| + c$

9) $\int \sqrt{a^2+x^2} dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} log\left|x + \sqrt{x^2+a^2}\right| + c$

These formulae can be applied only after reducing the quadratic function

$f(x) = ax^2 + bx + c$ to the canonical form $f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$.

Here we suggest an alternative set of formulae in a new format that reduces the number of steps required for simplification, and thus the burden of the process, helping the student to identify the correct answer more quickly. This paper shows how to greatly shorten time spent on such questions by using principles associated with Vedic Mathematics.

Methods

Quadratic Equation: quadratic forms always have derivatives and discriminants as in,

$f(x) = ax^2 + bx + c$, with derivative $f'(x) = 2ax + b$ and discriminant, $D = b^2 - 4ac$.

The Vedic method derives the first differential of each term by multiplying its Ghata (the exponent) by the Anka (its coefficient) and reducing power of x by one. A second observation is that $2ax + b = \pm\sqrt{b^2 - 4ac}$ i.e. $f' = \pm\sqrt{D}$ (the derivative of a quadratic form is equal to the square root of the discriminant) [4]. These two observations allow us to express results concerning integrals by replacing $2ax + b$ by f', and $b^2 - 4ac$ by D, as demonstrated in the worked examples below. Both derivative and discriminant can be calculated mentally without using pen and paper, and the final answer obtained directly by referring to these results.

Results

Simplified versions of six different types of quadratic integral are presented below. The number of steps using the conventional method and the Vedic short cut method are compared in Table 1. Comparative lengths of calculations are illustrated in Table 2.

Type 1:

$$I_1 = \int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{D}} \log\left[\frac{f' - \sqrt{D}}{f' + \sqrt{D}}\right], & a > 0, D > 0 \\ \frac{1}{\sqrt{D}} \log\left[\frac{\sqrt{D} - f'}{\sqrt{D} + f'}\right], & a < 0, D > 0 \\ \frac{2}{\sqrt{|D|}} \tan^{-1}\left(\frac{f'}{\sqrt{|D|}}\right), & a > 0, D < 0 \\ \frac{2}{f'}, & D = 0 \end{cases} \quad (1)$$

Type 2:

$$I_2 = \int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \log\left[\frac{f'}{2a} + \sqrt{\frac{f}{a}}\right], & a > 0, D > 0 \\ \frac{1}{\sqrt{|a|}} \sin^{-1}\left[\frac{|f'|}{\sqrt{D}}\right], & a < 0, D > 0 \\ \frac{1}{\sqrt{a}} \sinh^{-1}\left[\frac{f'}{\sqrt{D}}\right], & a > 0, D < 0 \end{cases} \quad (2)$$

Type 3:

$$I_3 = \int \frac{px + q}{ax^2 + bx + c} dx = l.\log f + m.I_1, \quad (3)$$

where $l = \frac{p}{2a}, m = q - lb$

Type 4:

$$I_4 = \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx = l.2\sqrt{f} + m.I_2, \quad (4)$$

where $l = \frac{p}{2a}, m = q - lb$

Illustrative examples comparing Vedic Maths method and conventional methods are discussed in table 2.

Discussion

The above problem solving methods emerging from observations of Vedic Mathematics incorporate short cuts allowing fast mental calculations. The new format using derivative and discriminant enables students to quickly select correct options for four line calculations. As can be seen, it greatly reduces the simplification process, increasing students' speed of calculation.

These patterns need to be presented in the right way and the right time, otherwise they do not appear to simplify what has to be done to any great extent. They are particularly appreciated when they are explicitly seen to reduce the labour involved in the normal procedures.

The methods may also be used to shorten algorithms in calculator and other applications requiring answers to quadratic integrals. We find that they greatly improve the speed and increase the confidence of students taking competitive professional exams, considerably improving average exam results e.g. mean improvements over 15%, lowest marks. Details of examination results obtained by students using these methods are being reported elsewhere.

Conclusion

This method of teaching students to recognize algebraic patterns helps stimulate student interest and enjoyment when teaching Integral Calculus. They also increase student confidence in working examples, and correspondingly improve level of performance on examples, tests etc. Thus, the new quadratic integral formats, listed in terms of derivative and discriminant, can help both students and teachers improve their speed of simplification. They have proved efficient, short cut methods of enhancing student confidence and competence.

Refernces

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Table 1: Conventional and Short Cut Methods

Comparison of Number of Steps

Conventional Method	Short Cut Method
Write the quadratic equation in complete square format	Recognize the sign of coefficient of x^2 and sign of Discriminant.
Recognize the sign of coefficient of x^2 and sign of a^2	Use the standard result to get the final answer.
Use the standard result to get the final answer	

Table 2: Illustrative Calculations for the four Types of Quadratic Integrals:

The table compares Conventional and Short cut methods with 4 illustrations. Mental calculations needed for the shortcut method are shown in the third column.

Type 1 - Illustration:

1. $\int \frac{\boldsymbol{dx}}{\boldsymbol{9x^2-12x+8}} =$

(a) $\frac{1}{6}\tan^{-1}\frac{3x+2}{2}$ (b) $\frac{1}{6}\log\frac{3x-2}{3x+2}$ (c) $\frac{1}{6}\sin^{-1}\frac{3x+2}{2}$ (d) $\frac{1}{6}\tan^{-1}\frac{3x-2}{2}$

Conventional Maths solution	*Short cut solution*	*Mental calculation*
$\int \frac{dx}{9x^2-12x+8}$ $=\int \frac{dx}{9\{x^2-\frac{12}{9}x+\frac{8}{9}\}}$ $=\frac{1}{9}\int \frac{dx}{\{x^2-\frac{4}{3}x+\frac{8}{9}\}}$ $=\frac{1}{9}\int \frac{dx}{\{x^2-2\left(\frac{2}{3}\right)x+\frac{4}{9}-\frac{4}{9}+\frac{8}{9}\}}$ $=\frac{1}{9}\int \frac{dx}{\left(x-\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2}$ $=\frac{1}{9}\frac{1}{\left(\frac{2}{3}\right)}\tan^{-1}\frac{x-\frac{2}{3}}{\frac{2}{3}}$ $=\frac{1}{6}\tan^{-1}\frac{3x-2}{2}$	$\int \frac{dx}{9x^2-12x+8}$ $=\frac{2}{\sqrt{144}}\tan^{-1}\frac{18x-12}{12}$ $=\frac{1}{6}\tan^{-1}\frac{3x-2}{2}$	Calculate $D = 144 - 288$ $= -144 < 0$ Recognize sign of a $a = 9 > 0$ $f' = 18x - 12$

Type 2 - Illustration:

2. $\int \frac{\boldsymbol{dx}}{\sqrt{\boldsymbol{1-4x-2x^2}}} =$

(a) $\frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{3}}{\sqrt{2}}(2x+1))$ (b) $\frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{2}}{\sqrt{3}}(2x+1))$

(c) $\frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{3}}{\sqrt{2}}(x+1))$ (d) $\frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{2}}{\sqrt{3}}(x+1))$

Conventional Maths solution	*Short cut solution*	*Mental calculation*
$\int \frac{dx}{\sqrt{1-4x-2x^2}}$ $= \int \frac{dx}{\sqrt{-2\left(x^2+2x-\frac{1}{2}\right)}}$ $= \int \frac{dx}{\sqrt{-2\left(x^2+2x+1-\frac{1}{2}-1\right)}}$ $= \int \frac{dx}{\sqrt{-2\left((x+1)^2-\frac{3}{2}\right)}}$ $= \int \frac{dx}{\sqrt{2\left(\frac{3}{2}-(x+1)^2\right)}}$ $= \int \frac{dx}{\sqrt{2}\sqrt{\left(\left(\frac{\sqrt{3}}{2}\right)^2-(x+1)^2\right)}}$ $= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x+1}{\frac{\sqrt{3}}{\sqrt{2}}} + \boldsymbol{c}$ $= \frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{2}}{\sqrt{3}}(x+1)) + \boldsymbol{c}$	$\int \frac{dx}{\sqrt{1-4x-2x^2}}$ $= \frac{1}{\sqrt{2}}\sin^{-1}\frac{4x+4}{\sqrt{24}} + c$ $= \frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{2}}{\sqrt{3}}(x+1)) + c$	Calculate $D = 16 - 4(-2)(1)$ $= 24 > 0$ Recognize sign of $a = -2 < 0$ $f' = -4x - 4$

Type 3 - Illustration:

3. $I = \int \frac{3x+1}{2x^2-2x+3}\boldsymbol{dx} =$

(a) $I = \frac{3}{4}\log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$

(b) $I = \frac{3}{4}\log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{4}\tan^{-1}\left(\frac{-2x-1}{\sqrt{5}}\right) + c$

(c) $I = \frac{3}{4}(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$

(d) $I = \frac{3}{4}log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-4}{\sqrt{5}}\right) + c$

Conventional Maths solution	*Short cut solution*	*Mental calculation*
$I = \int \frac{3x+1}{2x^2-2x+3}dx$ Let $3x + 1 = l(4x-2) + m$ $\Rightarrow 4l = 3 \Rightarrow l = \frac{3}{4},$ $-2l + m = 1$ $\Rightarrow m = 1 + \frac{3}{2} = \frac{5}{2}$ $Now\ I = \int \frac{\frac{3}{4}(4x-2)+\frac{5}{2}}{2x^2-2x+3}dx$ $I = \frac{3}{4}\int \frac{(4x-2)}{2x^2-2x+3}dx + \frac{5}{2}\int \frac{(4x-2)}{2x^2-2x+3}dx$ $= \frac{3}{4}\log(2x^2 - 2x + 3) + \frac{5}{2}I_1$ Consider $2x^2 - 2x + 3 = 2\left(x^2 - x + \frac{3}{2}\right)$	$I = \int \frac{3x+1}{2x^2-2x+3}dx$ $l = \frac{3}{4}, m = 1 - \frac{3}{4}(-2) = \frac{5}{2}$ $\therefore I = \frac{3}{4}\log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2}\frac{2}{\sqrt{20}}\tan^{-1}\left(\frac{4x-2}{\sqrt{20}}\right) + c$ $\therefore I = \frac{3}{4}\log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$	Calculate $D = 4 - 4(2)(3) = -20 < 0$ Recognize sign of $a = 2 > 0$ $f' = 4x - 2$

$= 2\left(x^2 - 2.\frac{1}{2}x + \frac{1}{4} + \frac{3}{2} - \frac{1}{4}\right)$ $= 2\left[\left(x - \frac{1}{2}\right)^2 + \frac{5}{4}\right]$ Now $I_1 = \frac{1}{2}\int \frac{dx}{\left[\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right]}$ $= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$ $\therefore I =$ $\frac{3}{4}\log(2x^2 - 2x + 3) + \frac{5}{2}\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$		
Type 4 - Illustration: 4. $\boldsymbol{I = \int \frac{3x+7}{\sqrt{1-x-x^2}}dx}$ (a) $I = -\sqrt{1-x-x^2} + \frac{11}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$ (b) $I = -\sqrt{1-x-x^2} + \frac{11}{2}\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c$ (c) $I = -3\sqrt{1-x-x^2} + \frac{11}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$ (d) $I = -3\sqrt{1-x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$		
Conventional Maths solution	*Short cut solution*	*Mental calculation*
$I = \int \frac{3x+7}{\sqrt{1-x-x^2}}dx$ Let $3x + 7 = l\frac{d}{dx}(1 - x - x^2) + m$ $\Rightarrow -2l = 3 \Rightarrow l = \frac{-3}{2}, -l + m = 7 \Rightarrow m = \frac{11}{2}$ $I = \int \frac{\frac{-3}{2}(-1-2x)+\frac{11}{2}}{\sqrt{1-x-x^2}}dx$ $\Rightarrow I = \int \frac{\frac{-3}{2}(-1-2x)}{\sqrt{1-x-x^2}}dx + \frac{11}{2}\int \frac{dx}{\sqrt{1-x-x^2}}$ Consider $1 - x - x^2 = -(x^2 + x - 1)$ $= -(x^2 + 2.\frac{1}{2}.x + \frac{1}{4} - \frac{1}{4} - 1)$ $= -\left[\left(x + \frac{1}{2}\right)^2 - \frac{5}{4}\right]$ $= \left[\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2\right]$ $\therefore I' = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}} = \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$ $\therefore I = -3\sqrt{1-x-x^2} + \frac{11}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$	$I = \int \frac{3x+7}{\sqrt{1-x-x^2}}dx$ $l = -\frac{3}{2},$ $m = 7 - -\frac{3}{2}(-1) = \frac{11}{2}$ $\therefore I =$ $-\frac{3}{2}2\sqrt{1-x-x^2} +$ $\frac{11}{2}.\frac{1}{\sqrt{1}}\sin^{-1}\left(\frac{\lvert -2x-1 \rvert}{\sqrt{5}}\right) + c$ $\therefore I = -3\sqrt{1-x-x^2} +$ $\frac{11}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$	Calculate $D = (-1)^2 - 4(-1)(1) = 1 + 4 = 5 > 0$ Recognize sign of $a = -1 < 0$ $f' = -2x - 1$