Bharati Krishna's Special Cases

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Abstract:

The numerous Special Cases given by Sri Bharati Krishna Tirthaji and his advocation of their use deserves greater study, not only as regards the many special cases that abound in mathematics but also in educational terms. In this paper these educational advantages are briefly described and three categories in which the special methods can be researched are suggested. Examples are given that show new special cases among these categories and the inclusion of trigonometric functions and derivatives of functions where Bharati Krishna uses only numbers or polynomials.

1. Introduction

Sri Bharati Krishna Tirthaji makes a clear and definite distinction (in his 'Author's Preface') between general and special cases.

"1. With regard to every subject dealt with in the Vedic Mathematical Sūtras, the rule generally holds good that the Sūtras have always provided for what may be termed the '*General Case*' (by means of simple processes which can be easily and readily – nay, instantaneously applied to any and every question which can possibly arise under any particular heading.

2. But, at the same time, we often come across special cases which, although classifiable under the general heading in question, yet present certain additional and typical characteristics which render them still easier to solve. And, therefore, special provision is found to have been made for such special cases by means of special Sūtras, sub-Sūtras, corollaries etc., relating and applicable to those particular types alone.

3. And all that the student of these Sūtras has to do is to look for the special characteristics in question, recognise the particular type before him and determine and apply the special formula prescribed therefor.

4. And, generally speaking it is only *in case* no special case is involved, that the general formula has to be resorted to. . . ." (Tirthaji, 1994)

These special cases, and the special methods that go with them, and also the general idea of special methods, are however not usually considered to be a part of modern mathematics teaching. Contemporary mathematics teaching relies mainly on general methods: and they are fine and very powerful as they handle a whole class of problems, but this can be like 'using a sledgehammer to crack a nut'.

Conventional mathematics does sometimes use special methods though, but they are not generally seen as such. For example, to multiply a whole number by 10 we simply attach a zero to the end of the number. We do not use long multiplication to multiply by 10 since we have this special device.

The Vedic system however takes this idea much further: we may know that some problem has a special characteristic and we can apply the special process required, which can be a huge saving in time. Examples are the method for multiplying numbers near a base or of subtracting a number from a power of 10. Multiplications like 8876 × 9997 and subtractions like 10,000 – 3456 are long and tedious by the usual methods, but noting the special types we can give the answer almost immediately with the Vedic Sutras.

Of course the special methods can be used or not: since there is a general method we can always use that. So we have the choice whether to learn and apply these special methods (or some of them) or not.

In everyday life we do not respond to every need for some food, for example, by contacting the local supermarket, we treat every problem on its own merits, use our intelligence and find the best solution for that particular situation. The skilled carpenter does not use a chisel for every job but selects the tool best suited to what needs to be done. So the use of special methods in mathematics is well-aligned with our everyday experience.

1.1 Educational Advantages

Adding this dimension to mathematics teaching could have considerable advantages.

The element of choice means the student can choose their method, or adapt one they know: instead of having to follow rigid rules they can make their own decisions and discover their own paths to solutions.

This opens up the possibility of mathematics being a creative subject, where students can have their own input. And this in turn means those pupils of a more creative or inventive disposition get more involved.

The special cases are so numerous that lessons can be structured around them and students can be encouraged to explore and innovate.

1.2 Three Areas of Research

Although a great deal of work has been done in developing and applying Bharati Krishna's methods the scope of the special methods has had little research even though it has tremendous potential. In this paper I would like to point out some ways in which the concept of Special Cases can be extended and developed.

Bharati Krishna shows many special methods in his book, and they fall beautifully under the various Sūtras. How far can these be developed, what other special methods are there, and do they all fall neatly under the Vedic Sūtras?

In fact we may distinguish three areas of research:

- Forms which may be restructured so that they fall within known special types.
- Exploring and expanding the range of application of the special methods Bharati Krishna has given.
- Looking for special methods in areas not touched by Bharati Krishna.

To this we may add a holistic look at special methods in general: what they show us about mathematics, the educational advantages of using this approach in schools and the possibility of useful general structures within the special methods.

We look now more closely at these three categories.

2. Restructuring a Problem in Order to Use a Known Special Type

Simple illustrations of this are:

1. Restructuring 88 × 197 into 88×97 + 88×100, i.e. bringing the problem within the special Nikhilam/base method by splitting 197 into 97+100.

2. Restructuring 88 × 196 into 88 × 98 × 2, i.e. bringing the problem within the special Nikhilam/base method by using proportion.

Bharati Krishna gives many examples of restructuring, which he amusingly describes as ‘disguised specimens’ (thin, medium and thick disguises).

2.1 When the Samuccaya is the Same, that Samuccaya is Zero

Bharati Krishna says this is a beautiful special Sutra and he gives a large number of variations, with different meanings of the word ‘*Samuccaya*’.

Example 1

Chapter XII of his book shows the special type: $\frac{1}{x-3}+\frac{1}{x-5}=\frac{1}{x-2}+\frac{1}{x-6}$. Here we note that the sum of the denominators on each side is the same, and so according to the Sutra that sum is zero:

$$2x - 8 = 0, \text{ and so } x = 4.$$

Example 2

A thinly disguised equation of this type would be $\frac{1}{x-3}-\frac{1}{x-6}=\frac{1}{x-2}-\frac{1}{x-5}$ because it can be easily rearranged into the previous form.

Example 3

A thicker disguise is illustrated by Bharati Krishna by $\frac{x-2}{x-3}+\frac{x-3}{x-4}=\frac{x-1}{x-2}+\frac{x-4}{x-5}$.

Here division in each of the 4 fractions gives: $1+\frac{1}{x-3}+1+\frac{1}{x-4}=1+\frac{1}{x-2}+1+\frac{1}{x-5}$ and when the two 1s are cancelled from each side we see that the sum of the denominators on each side is the same and so the answer is immediately available.

We may ask: “what other equations can be transformed to come under this (or any other) special type?”And what characteristics can we look out for to spot them?

2.2 Only the Last Terms

Chapter XVI (Third Type) shows another special type:

Example 4

$$\frac{3x^2+5x+8}{5x^2+6x+12}=\frac{3x+5}{5x+6}$$

This is of the form $\frac{AC+D}{BC+E}=\frac{A}{B}$ (where $A = 3x + 5$, $B = 5x + 6$ and $C = x$ in this example) and Bharati Krishna shows that this leads to $\frac{D}{E}=\frac{A}{B}$.

So, for this equation we can say: $\frac{3x+5}{5x+6}=\frac{8}{12}$, which leads to $x = 3$ and it is the only solution.

2.2.1 Proof and Extension of the Method

Bharati Krishna gives two proofs. The second of which is as follows.

$$\frac{AC+D}{BC+E}=\frac{A}{B} \;\therefore\; ABC+AE=ABC+BD,\;\therefore\; AE=BD,\;\therefore\; \frac{D}{E}=\frac{A}{B}.$$

However we note that although Bharati Krishna does only give examples where D and E are numbers, they need not be, as illustrated by the following example.

Example 5

$$\frac{x^2+3x-2}{x^2+3x}=\frac{x+3}{x+2}$$

Here we do not have the multiples of the RHS on the left.

However we can restructure the denominator on the LHS:

$$\frac{x^2+3x-2}{x^2+2x+x}=\frac{x+3}{x+2}$$

We therefore get $\frac{x+3}{x+2}=\frac{-2}{x}$ and this leads to the two solutions $x = -1, -4$.

This is a simple case of restructuring, but it suggests many extensions to the range of application of this type, and indeed of all special types, and is one of the areas for further study mentioned.

The restructuring involved here is a very simple one, but there are very many other examples that could be given in which this Sutra can be applied after a little restructuring.

3. Expanding the Range of Application of the Special Methods Given by Bharati Krishna

3.1 If One is in Ratio the Other One is Zero

Bharati Krishna shows a special type of simultaneous equations using the Sutra *If One is in Ratio the Other One is Zero* (Chapter XV).

Example 6

So for the simultaneous equations $6x + 7y = 8$

$$19x + 14y = 16,$$

we note that the y-coefficients are *in ratio* with the right-hand coefficients: 7:14 = 8:16.

This tells us that '*the Other One*', i.e. x, is zero. And so $x = 0$ (and $y = 8/7$).

Bharati Krishna shows some variations of this with three equations in three unknowns. Here is another variation:

Example 7

$$2x + 3y = 6$$

$$4x^2 + 5y = 36$$

We note here that $4x^2$ is the square of $2x$, just as the numbers on the right have the same relationship (36 is 6^2).

This indicates that the other variable is zero, so $y = 0$ (and $x = 3$).

It is fairly obvious that since $y = 0$ leads to a consistent result then $y = 0$ gives a solution.

This shows a different application of the Sutra, and when it is spotted can save time and effort.

3.2 Non-polynomial Functions

Returning now to the special type referred to earlier (see 2.2) under the Sutra *Only the Last Terms,*

we note that the proof shows that the terms A, B, C, D, E in $\frac{AC+D}{BC+E} = \frac{A}{B}$ could each be polynomial expressions. However we may further note that they could be other functions of x: trig functions , derivatives, etc.

Example 8

$$\frac{2\sin x + \cos x}{5\sin x + 2} = \frac{2}{5}$$

Here we see the same pattern with $A = 2$, $B = 5$ and $C = \sin x$.

Therefore $\frac{\cos x}{2} = \frac{2}{5}$ and so $\cos x = \frac{4}{5}$ (which gives all the solutions to the given equation).

Example 9

Consider $\frac{2y'+x+5}{3y'+6}=\frac{2}{3}$, where $y' = \frac{dy}{dx}$ and y is a function of x.

We note the same pattern with $A = 2$, $B = 3$ and $C = y'$.

Therefore $\frac{x+5}{6}=\frac{2}{3}$ and so $x = -1$ (the only solution to the original equation).

3.3 When the Samuccaya is the Same, that Samuccaya is Zero

Bharati Krishna's fourth meaning of the word '*Samuccaya*' is 'combination (or total)' and "in this sense, it is used in several different contexts".

Example 10

$$\frac{2x+9}{2x+7}=\frac{2x+7}{2x+9}$$

We note that the total of the numerators and the total of the denominators is the same. So, according to the Sutra that total can be equated to zero: $4x + 16 = 0$, and so $x = -4$.

Example 11

$$\frac{3x+4}{6x+7}=\frac{5x+6}{2x+3}$$

Here we get $8x + 10 = 0$ and so $x = -5/4$.

The previous examples of this type were in fact linear but here it is in fact a quadratic.

However Bharati Krishna points out that the same Sutra gives the other solution too: the difference of numerator and denominator is the same on both sides and we can equate this to zero: $3x + 3 = 0$, and $x = -1$.

These are quite striking special cases and there must be many forms that can be restructured to bring them under this method.

The form of the equations involved here is: $\frac{A}{B}=\frac{C}{D}$, where A and B are binomials and

$A + C = B + D$. Then a solution is $A + C = 0$.

Bharati Krishna does not supply a proof of this but it too can be extended to other functions. That is to say A, B, C and D can each be functions of x.

So, they can be quadratics, cubics, trig or exponential functions etc.

Example 12

$$\frac{2\sin x}{\sin x+\cos x}=\frac{\cos x}{\sin x}$$

Here we observe that the sum of the numerators equals the sum of the denominators. Therefore $2\sin x+\cos x=0$.

And the difference of numerator and denominator on each side is $\sin x - \cos x$ therefore $\sin x - \cos x = 0$.

The equations $2\sin x + \cos x = 0$ and $\sin x - \cos x = 0$ provide all the solutions of the given equation.

Example 13

$$\frac{2y'+1}{2y'} = \frac{3x}{3x+1}$$

Again we see that the sum of the numerators = the sum of the denominators = $2y'+1+3x$.

Therefore $2y'+1+3x=0$.

Taking differences of numerator and denominator here does not lead to any further solutions, and in fact $2y'+1+3x=0$ gives all the solutions to the given equation.

Example 14

$$\frac{y'+1}{6-x} = \frac{5-x}{y'}$$

Here we find that $y'+6-x$ is both the sum of the numerators and of the denominators.
So $y'+6-x=0$. [1]

In this case the difference on each side is $y'+x-5$, so $y'+x-5=0$. [2]

Equations [1] and [2] lead to all the solutions of the given equation.

4. Special Methods in Areas not Touched by Bharati Krishna

These are applications of the Sutras in which some special characteristic is noted that leads to the application of one of the Sutras.

4.1 If One is in Ratio the Other One is Zero

Example 15

Find a vector perpendicular to the vectors $a = -3i + 2j + 7k$
$b = 6i - 4j + k$

We note here that since the coefficients of $-3i + 2j$ and $6i - 4j$ are *in ratio* the required coefficient of k is zero. Therefore $2i + 3j$ will serve as the required perpendicular vector (since $2i + 3j$ is perpendicular to $-3i + 2j$)

4.2 Special Caes of trigonometric Equations

Under the same *Samuccaya* Sutra mentioned before but with a different meaning of the term *Samuccaya* there are some special trigonometric equations that are easily solved.

Example 16

$$\cos 4A + \cos 2A = \cos A$$

In this equation we note that since:

the multiple of *A* on the RHS is half the difference of the multiples on the LHS, then the RHS is zero,

that is, $\cos A = 0$ gives a set of solutions to the given equation.

There is another set of solutions though and these are given by $\cos 3A = \frac{1}{2}$ where the 3*A* is half the sum of the multiples of *A* on the LHS.

$\cos A = 0$ and $\cos 3A = \frac{1}{2}$ lead to all the solutions of the given equation.

Example 17

Similarly given $\cos 7A + \cos 3A = \cos 2A$ we note that 2*A* is half the difference of 7*A* and 3*A* therefore $\cos 2A = 0$ (and $\cos 5A = \frac{1}{2}$).

The meaning of the term *Samuccaya* here will be "cosine of half the difference".

Example 18

$$\cos 4A + \cos 2A = \cos 3A$$

Note here that the LHS is the same as in Example 16, but now we have the cosine of half the sum on the right: ½(4*A* + 2*A*) = 3*A*.

Again we can say that RHS = 0, i.e. $\cos 3A = 0$.

And the other set of solutions is given by $\cos A = \frac{1}{2}$ where the multiple of *A* here is half the difference of those on the LHS: ½(4*A* – 2*A*) = *A*.

Example 19

$$\cos 6A + \cos 20A = 5\cos 13A$$

Here we see that 13 is half the sum of 6 and 20 and so $\cos 13A = 0$.

That is, the extra factor, 5, makes no difference: we just equate the RHS to zero.

(The other set of solutions is $\cos 7A = \frac{5}{2}$, where 7 is half the difference of 6 and 20, and 5 is the coefficient on the RHS of the given equation.)

These examples show two of the 16 types of trig equations that can be solved this way.

4.2.1 Proof using Triples

Triples (Williams 2010) can be used to prove that if,

$$\cos P + \cos Q = \cos\left(\frac{P \pm Q}{2}\right) \text{ then } \cos\left(\frac{P \pm Q}{2}\right) = 0$$

X	a	b	1	
Y	c	d	1	$\pm$
$X+Y$	$ac-bd$	$bc+ad$	1	
$X-Y$	$ac+bd$	$bc-ad$	1	

Above is shown the addition and subtraction of two general triples where $X, Y \in \mathbb{N}$ are angles.

First note that looking at the angles (in the 1st column), X is half the sum of the bottom two angles, and Y is half the difference of the bottom two angles.

Also note that in the 2nd column the sum of the bottom two elements is twice the product of the top two elements. That is:

$$\cos(X+Y)+\cos(X-Y)=2\cos X\cos Y.$$

This is a general result, but it means that if we have a particular equation in which we have a sum of two cosines, on the LHS say, equal to the cosine of either half the sum or half the difference of the angles on the LHS then we have $\cos(X+Y)+\cos(X-Y)=\cos X$ or $\cos Y$, and therefore $2\cos X\cos Y=\cos X$ or $\cos Y$.

So in both cases we see that the RHS = 0 is a solution: $\cos X=0$ or $\cos Y=0$ respectively.

And the other set of solutions is $\cos Y=\frac{1}{2}$ or $\cos X=\frac{1}{2}$ resp.

Similarly, if we have $\cos(X+Y)+\cos(X-Y)=n\cos X$

or $\cos(X+Y)+\cos(X-Y)=n\cos Y$

then we get $2\cos X\cos Y=n\cos X$ or $2\cos X\cos Y=n\cos Y$

and then $\cos X=0$ or $\cos Y=0$ is still a solution. The other solutions being given by $\cos Y=\frac{n}{2}$ or $\cos X=\frac{n}{2}$ respectively.

* For the reader who is unfamiliar with triple addition and subtraction note that $X \mid a, b, 1$ is called a triple in which $a=\cos X$ and $b=\sin X$, and similarly for the other angles. The bottom two lines in the given chart follow immediately from the standard expansions for $\cos(A\pm B)$ and $\sin(A\pm B)$.

5. Conclusion

Bharati Krishna gives dozens of special cases and only a few are mentioned here. It is hoped that the nine or so new variations and applications shown in this paper will encourage others to further explore this fruitful aspect of Vedic mathematics: fruitful in terms of new research possibilities and also as an educational tool for use in schools.

The quote at the beginning of this paper ended with:

“And, generally speaking it is only *in case* no special case is involved, that the general formula has to be resorted to.”

This intriguing statement suggests that the initial mental preference when looking at a mathematics problem should be to inquire about a special case *before* considering the general case.

In this way perhaps we can change the reaction of a child to a mathematics question from "what algorithm am I supposed to apply?" to the more intelligent "let me look closely to see what I can find out".

References

[1] Bharati Krishna Tirthaji Maharaja, (1994). Vedic Mathematics. Delhi: Motilal Banarasidas,.

[2] Williams. K. R. (2010). Triples. U.K.: Inspiration Books.